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Aggregation as a Problem in Decision-Making under Ignorance or Uncertainty

by

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1. A large class of aggregation problems seems to reduce to the type of situation described below.

Let the outcome $\omega \in \Omega$ of the decision-maker's decision $d \in D$ depend on an event $e \in E$ ("state of nature") over which he has no control. (The set Ω is arbitrary.) Thus

$$\omega = \bar{\bar{\omega}}(d, e).$$

Let

$$w = \bar{\bar{w}}(\omega)$$

be the magnitude the decision-maker is seeking to maximize ("utility").

We shall assume that no element of D is inadmissible. To each d we assign a "value" v_d. In general,

$$v_d = \bar{\bar{v}}_d(\bar{\bar{\omega}}).$$

The function $\bar{\bar{v}}_d$ is the "decision-criterion." The decision-maker is assumed to be selecting d so as to maximize v_d.

The decision-maker's strategic domain D will now be augmented in the following fashion. Denote by $\mathcal{P}_\alpha$ $(\alpha \in \alpha)$ a partitioning of E. [I.e., $\mathcal{P}_\alpha$ represents a class of subsets $\{E_\alpha^\beta; \beta \in \mathcal{B}\}$ with $E_\alpha^{\beta'} \cap E_\alpha^{\beta''} = 0$ for $\beta' \neq \beta''$ and the union $U_{\beta \in \beta}\ E_\alpha^\beta = E.$] Also, define a set Γ such that

$$\gamma_\alpha = \bar{\gamma}(\alpha) \qquad . \quad \gamma_\alpha \in \Gamma.$$

[I.e., to each partition $\mathcal{P}_\alpha$ of E corresponds a unique element γ_α in Γ. γ_α is the "real cost" of $\mathcal{P}_\alpha$.]

The partitioning, say $\mathcal{P}_\alpha$, of E has two consequences. First, the "utility" w associated with a given decision d depends on the α chosen. Thus we now have

$$w = \bar{\varphi}[\bar{\omega}'(d, e), \gamma_\alpha]$$

Second, additional strategies are available to the decision-maker. The "game" proceeds as follows: 1. the decision-maker selects α, say α_o; 2. the decision-maker is told in which set of the partition $\mathcal{P}_{\alpha_o}$ the true e is to be found (e.g., he is told that $e \in E_{\alpha_o}^{\beta_o}$); 3. the decision-maker selects the optimal decision, say d_o. Hence, corresponding to a given α_o, the decision-maker may use any of the strategies which to each $E_{\alpha_o}^{\beta}$ assign a unique element $d_\beta \in D$, i.e., any of the (decision) functions

$$d_\beta = \bar{d}_{\alpha_o}\ (E_{\alpha_o}^{\beta}).$$

[Thus the decision-maker chooses an element $\bar{d} \in \bar{D}$ where $\bar{D}$ is a class of decision functions; the $d \in D$ are still the "terminal decisions."]

Clearly, in terms of ω it is to the decision-maker's advantage to use as fine a partitioning as possible. ($\mathcal{P}_{\alpha'}$ is said to be "finer" than $\mathcal{P}_{\alpha''}$ if each $E_{\alpha''}^{\beta}$ is a union of the $E_{\alpha'}^{\beta}$, while the converse is not true. The "finest" partitioning $\mathcal{P}_1$ assigns only one element of E to each of the E_1^{β}. The "coarsest" partitioning $\mathcal{P}_o$ consists of only one set $E_o^1 = E$. "Fineness" is a partial ordering on $\{\mathcal{P}_\alpha ; \alpha \in \mathcal{A}\}$.) However, we may expect the "cost" γ_α of finer partitioning to be higher in utility terms. Hence, other things being equal, coarser or finer partitioning may be found to be optimal depending on the nature of $\bar{\gamma}$.

2. Example

Let $E = (e_1, e_2)$, $D = (d_1, d_2)$. There are only two ways to partition E:

$$\mathcal{P}_1: \ E_1^1 = (e_1),\ E_1^2 = (e_2)$$

$$\mathcal{P}_0: \ E_0^1 = (e_1, e_2).$$

The complete set $\bar{\bar{D}}$ of decision-functions is:

$\bar{\bar{d}}_{0,11}$: choose $\mathcal{P}_0$ and then d_1;

$\bar{\bar{d}}_{0,22}$: choose $\mathcal{P}_0$ and then d_2;

$\bar{\bar{d}}_{1,11}$: choose $\mathcal{P}_1$ and then d_1;

$\bar{\bar{d}}_{1,12}$: choose $\mathcal{P}_1$ and then $\begin{cases} d_1 \text{ if } E_1^1 \\ d_2 \text{ if } E_1^2 \end{cases}$;

$\bar{\bar{d}}_{1,21}$: choose $\mathcal{P}_1$ and then $\begin{cases} d_2 \text{ if } E_1^1 \\ d_1 \text{ if } E_1^2 \end{cases}$;

$\bar{\bar{d}}_{1,22}$: choose $\mathcal{P}_1$ and then d_2.

Assuming that $w = \bar{\bar{\varphi}}(\omega, \gamma_0) > \bar{\bar{\varphi}}(\omega, \gamma_1)$ for all ω, we see that $\bar{\bar{d}}_{1,11}$ and $\bar{\bar{d}}_{1,22}$ are inadmissible since they waste costly information. Hence, writing $\omega_{ij} = \bar{\bar{\omega}}(d_i, e_j)$ we obtain the "payoff matrix" (with the two inadmissible strategies eliminated):

$\bar{\bar{D}}$ \ E	e_1	e_2
$\bar{\bar{d}}_{0,11}$	$\bar{\bar{\varphi}}(\omega_{11}, \gamma_0)$	$\bar{\bar{\varphi}}(\omega_{12}, \gamma_0)$
$\bar{\bar{d}}_{0,22}$	$\bar{\bar{\varphi}}(\omega_{21}, \gamma_0)$	$\bar{\bar{\varphi}}(\omega_{22}, \gamma_0)$
$\bar{\bar{d}}_{1,12}$	$\bar{\bar{\varphi}}(\omega_{11}, \gamma_1)$	$\bar{\bar{\varphi}}(\omega_{22}, \gamma_1)$
$\bar{\bar{d}}_{1,21}$	$\bar{\bar{\varphi}}(\omega_{21}, \gamma_1)$	$\bar{\bar{\varphi}}(\omega_{12}, \gamma_1)$

Numerically (in dollars), let

$$\|\omega_{ij}\| = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \quad \text{(the decision-maker's income)}$$

$\gamma_0 = 0$

$\gamma_1 = -.2$ (payment by the decision-maker.)

Then the payoff matrix (assuming utility linear in income) is

$\bar{D}$ \ E	e_1	e_2
$\bar{d}_{0,11}$	0	2
$\bar{d}_{0,22}$	1	0
$\bar{d}_{1,12}$	-.2	-.2
$\bar{d}_{1,21}$	.8	1.8

Clearly in this case $\bar{d}_{1,12}$ is also inadmissible. As for the other three strategies, by such decision criteria as the "maximin" or the "mid-point of maximin and maximax*" $\bar{d}_{1,21}$ is optimal. (According to the "maximax" criterion, however, $\bar{d}_{0,11}$ is optimal.) Had the cost of partitioning $\mathcal{P}_1$ been much higher (in utility terms), one of the other two strategies (i.e., $\mathcal{P}_0$) might have been found optimal.

3.1. In problems of aggregation we usually deal with a multi-dimensional variable $x \in X$ (e.g., $\underline{x}$ is the n-dimensional vector of all outputs with $\underline{n}$ very large). Since it is found unmanageable to deal with such large $\underline{n}$, a vector of $\xi \in \Xi$ of lower dimensionality (say $\underline{m}$ much lower than $\underline{n}$) is obtained with the help of a functional relation F so that

$$\xi = F\ (x).$$

The problem is to find an F optimal in $\mathcal{F}$, say $\hat{F}_{\mathcal{F}}$, where it is specified that we must have $F \in \mathcal{F}$. [A simple example is the following: $\mathcal{F}$ is defined by the condition that ξ must be one-dimensional and $\xi = \theta\, y_1 + (1 - \theta)\, y_2$, $0 \leq \theta \leq 1$ where $y_i = g_i(x)$ are scalars and the g_i are given.]

3.2. Now we establish correspondence between the problem (incompletely) stated in § 3.1 and the language of § 1.

X corresponds to E, $\underline{x}$ to $\underline{e}$. A given F corresponds to a given partitioning $\mathcal{P}_\alpha$. To the set E_α^β (for a given pair α, β) corresponds the inverse image

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* I.e., $S(\frac{1}{2})$ in Cowles Commission Discussion Paper, Statistics No. 356, § 4.

in X (for a given F) of a given point ξ in Ξ. To ζ corresponds the cost of collecting and manipulating data (ξ) of a degree of aggregation determined by the F chosen.

In order to obtain the counterpart of ω and a decision problem we must assume that someone will be making a decision whose consequences depend, among others, on the true value of $\underline{x}$. The more is known about $\underline{x}$ (i.e., about a set of which $\underline{x}$ is an element) the easier it is to make a "good" decision. But information about $\underline{x}$ is gained only through observations on ξ. Generally speaking the more "informative" ξ is (m larger, etc.), the higher the cost.

3.3. In practice the situation is, of course, much more complicated. In particular, an aggregation formula (an F) must be selected without precise knowledge of the decision problem it is likely to be used in. Hence, presumably, one should optimize over a class of such problems. (A priori probabilities may be available here.) The formal apparatus of decision-making under ignorance may still be used here*.

4. Something may be known about the ("a priori") probability distribution on E (i.e., on X, in terms of ξ3). In that case the approach indicated in Cowles Commission Discussion Paper, Statistics No. 355 may be used.

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* Cf., Leonid Hurwicz, "Theory of Economic Organization", Econometrica, Vol. 19, No. 1, p. 54, January 1951.