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# Distribution of Ratios of Quadratic Forms.[1]

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## 1. Introduction.

This paper considers the problem of finding the distribution function of a ratio of quadratic forms in normally distributed random variables. Special emphasis will be given to some particular quadratic forms arising in the theory of serial correlation. As the reader will see (or perhaps has seen) this problem is highly entangled with hyperelliptic integrals; and the strategy which evolves in coping with it is to by-pass these integrals by obtaining approximations to the exact distribution which are correct within an arbitrary degree of accuracy. It is apparent upon perusal of the literature on serial correlation, quadratic forms, and ratios of quadratic forms, that approximations of one sort or another play a very important role, since the exact solutions, even when available, are too difficult to apply in reality. The introduction of circularity by Hotelling (cf. R. L. Anderson [1]), the smoothing of the characteristic function by Koopmans [2], the series expansion of Kent used by Hart [3] in computing the distribution of von Neumann's statistic [4], the series expansion for the distribution of quadratic forms and ratios of quadratic forms given by Robbins [5], [6], are a few instances of approximative methods.

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1. The author is grateful to L. Hurwicz for his keen interest and encouragement.

The present article discusses a method of reducing the general problem of the distribution of ratios of quadratic forms to a much simpler problem, and suggests possible methods of obtaining suitable approximations in the light of this simplified approach. The geometrical method in § 4, although stated in the context of the least-squares statistic of Hurwicz [7], may be applied generally for the case of $n=3$ observations. Its amenability to extension for $n > 3$, however, is problematical. As for the method of Gram-Charlier series discussed in § 6, it has the advantage of comparative simplicity, but it requires justification on exact grounds before it is applicable. (See supplement for discussion of this and other questions).

2. Distribution by means of Fourier inversion formulae.

Let the random variable $X = (X_1, X_2, \ldots, X_n)$ have the probability density

$$p(x) = \frac{(\det \Sigma)^{1/2}}{2\pi^{n/2}} e^{-\frac{1}{2} x \Sigma x'} \tag{1}$$

where x is the row vector $(x, x_1, \ldots, x_n)$ and $x'$ the corresponding column vector.[2] $x \Sigma x'$ is positive definite. The distribution of

$$\frac{X Q X'}{X P X'} \tag{2}$$

will be considered, where P, Q are arbitrary symmetric $n \times n$ matrices. The joint characteristic function of $X Q X'$ and $X P X'$ is

$$\phi(t_1, t_2) = E(e^{it_1 xQx' + it_2 xPx'}) = \left[\frac{\det \Sigma}{\det(\Sigma - 2it_1 Q - 2it_2 P)}\right]^{1/2} . \tag{3}$$

Denoting the distribution function of (2) by

$$G(\xi) = P\left\{\frac{X Q X'}{X P X'} \leq \xi\right\} . \tag{4}$$

it follows from Gurland [8] that

$$G(\xi) + G(\xi - 0) = 1 - \frac{1}{\pi i} \oint \frac{\phi(t, -t\xi)}{t} dt \tag{5}$$

2. We adopt the convention of denoting a random variable by a capital letter, while reserving the corresponding small letter for a particular sure value.

provided $P\left\{X P X' \leq 0\right\} = 0$. The notation $⨍$ signifies the Cauchy principal value $\lim_{\substack{\varepsilon \to 0 \\ T \to \infty}} \left( \int_{-T}^{-\varepsilon} + \int_{\varepsilon}^{T} \right)$. It is also possible to give the corresponding inversion for the case $P\left\{X P X' \leq 0\right\} > 0$, (cf. Gurland [8]), but this will not be considered here.

The probability density $G'(\xi)$ obtains on differentiation of (5) and is given by

$$G'(\xi) = \frac{1}{2\pi i} ⨍ \left[ \frac{\partial \phi(t_1, t_2)}{\partial t_2} \right]_{t_2 = -t_1 \xi} dt_2 \tag{6}$$

in every interval where this integral converges uniformly. It is evident from (3) that in general, for $n \geq 3$, the integrals in (5) or (6) will be of elliptic or hyper-elliptic type (cf. Goursat [9]), unless the determinant

$$\det(\Sigma - 2it\, Q + 2it\, \xi\, P) \tag{7}$$

has n equal factors, or has all its factors equal in pairs. An instance where the factors are pairwise equal occurs when (2) is the circular serial correlation coefficient with

$$Q = \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{Vmatrix} \tag{8}$$

and

$$P = \Sigma = I_n \tag{9}$$

where $I_n$ represents the unit matrix of order n. As a consequence, the exact distribution of the circular coefficient is greatly simplified, and it has been obtained by Anderson [1], but by a different method from the one described above.

An instance where the factors of (7) are all distinct is given by (9) and

$$Q = \begin{Vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{Vmatrix} \qquad (10)$$

This refers to the noncircular serial correlation coefficient, and its distribution is given by Koopmans [2] in terms of a contour integral and by von Neumann [4] in terms of the $\left(\frac{n}{2} - 1\right)$st derivative (n even) of the probability density.

It is interesting to note that while the distribution of the noncircular statistic above involves hyperelliptic integrals, its moments are obtainable (cf. von Neumann [4], Williams [10]) from easily evaluated integrals which are not hyperelliptic. In the general case, $(\Sigma \neq I_n)$ however, the situation is much worse. Both the distribution and its moments involve hyperelliptic integrals. The only results, so far, in this direction, are due to Hurwicz [7], who evaluates the first moment of (2) for small values of n, when Q is given by (10) and

$$P = \begin{Vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{Vmatrix} \qquad (11)$$

$$\Sigma = \left\| \begin{array}{cccccccccc} 1 & -\alpha & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ -\alpha & 1+\alpha^2 & -\alpha & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & -\alpha & 1+\alpha^2 & -\alpha & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ 0 & 0 & 0 & & & 0 & -\alpha & 1+\alpha^2 & & -\alpha \\ 0 & \cdot\, 0 & \cdot\,\cdot\, 0 & \cdot\, 0 & \cdot\,\cdot\,\cdot\,\cdot & 0 & 0 & -\alpha & & 1 \end{array} \right\| \tag{12}$$

For larger values of $n$ (hyperelliptic case) he investigates the behavior of the first moment by means of Taylor expansions in the parameter.

Before proceeding to § 3, it should be remarked that for certain forms of Q, P, $\Sigma$ and for some values of $\xi$, it might be possible to evaluate integrals such as (5) or (6) in finite terms (cf. Ritt [11],[3] that is, in terms of a finite number of logarithmic and irrational functions. Goursat [9] gives conditions for an elliptic integral to be pseudo-elliptic.

3. Distribution problem from a simpler viewpoint.

It will be shown here that the problem of finding the distribution of the ratio (2), where X has probability density (1), Q, P are symmetric, and $P\left\{X P X' \leq 0\right\} = 0$ is reducible to the following simpler problem: To find the probability

$$P\left\{X \Lambda X' \leq 0\right\} \tag{13}$$

where X has the probability density

$$p(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2} x x'} \tag{14}$$

and $\Lambda$ is a diagonal matrix the elements of which are specified constants $\lambda_1, \lambda_2, \ldots, \lambda_n$.

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3. The author is indebted to Morton Slater for this reference.

This reduction is actually inherent in the inversion formula (5) but it is clarifying to emphasize it here.

For any real z, let

$$R_z = Q - z P \tag{15}$$

and

$$H_z(\xi) = P\left\{X R_z X' \leq \xi\right\} . \tag{16}$$

Then $$H_z(0) = P\left\{\frac{X Q X'}{X P X'} \leq z\right\} = G(z). \tag{17}$$

Let $\lambda_i(z)$ $(i = 1, 2, \ldots, n)$ be the roots of

$$\det(R_z - \lambda \Sigma) = 0 \tag{18}$$

and $\Lambda_z$ be a diagonal matrix with these roots as the diagonal elements. As $x \Sigma x'$ is positive definite, the $\lambda_i(z)$ are all real; and there exists a matrix T such that (cf. Bocher [12])

$$T \Sigma T' = I_n \tag{19}$$

and

$$T R_z T' = \Lambda_z \quad . \tag{20}$$

Hence

$$H_z(\xi) = P\left\{X \Lambda_z X' \leq \xi\right\} \quad \text{where}$$

X now has the probability density of (14). Now $X \Lambda_z X'$ may be interpreted as a linear combination of independent random variables each of which is distributed as $\chi^2$ with one degree of freedom. The problem of finding G(z) reduces to finding the distribution function of this linear combination <u>at the one point $\xi = 0$</u>.

In the geometrical language of the sample space we may say the problem is reduced to finding the probability measure of the interior of the cone

$$x \Lambda_z x' = 0 \tag{21}$$

when the probability density is given by (14). If $z_0$ is the largest value of z for which all $\lambda_i(z) \geq 0$, and $z_1$ is the smallest value of z for which $\lambda_i(z) \leq 0$, then the range of (2) is the interval $(z_o, z_1)$. Of course, $z_o$, $z_1$ need not be finite.

It should be noted that the $\lambda_i(z)$ are obtainable on factoring the determinant (7), for the characteristic function is uniquely determined. Hence

$$\prod_{j=1}^{n} (1 - 2i \ \lambda_j(z) \ t) = k \ \det(\Sigma - 2it \ Q + 2it \ z \ P) \tag{22}$$

where k is the appropriate normalizing constant.

Even if the $\lambda_j(z)$ do not follow a simple pattern, such as for instance in serial correlation, the moments of $X \wedge_z X'$ may be found exactly, without knowledge of the $\lambda_j(z)$, by differentiation of the characteristic function $\phi(t, - tz)$ at the point $t = 0$. By means of these moments it might be possible to approximate $H_z(0)$ with predetermined accuracy. This question is discussed in §6.

4. Distribution of the least-squares statistic of Hurwicz.

For Q, P, Σ defined respectively by (10), (11), (12), the characteristic function is (cf. Hurwicz [7])

$$\sqrt{1 - \alpha^2}\ \phi^{-2}(t_1, t_2) = \det \begin{Vmatrix} y & f & 0 & 0 & \cdots & 0 \\ 1 & z & f & 0 & \cdots & 0 \\ 0 & 1 & z & f & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots 1 \ z & f \\ 0 & 0 & 0 & 0 & \cdots 0 \ 1 & 1 \end{Vmatrix} = \det C^{(n)}(t_1, t_2), \text{ say} \tag{23}$$

where

$$\begin{aligned} y &= -2it_2 + 1 \\ f &= (it_1 + \alpha)^2 \\ z &= -2it_2 + 1 + \alpha^2. \end{aligned} \tag{24}$$

It is clear that $\det C^{(n)}(t, -t\xi)$ is a polynomial of degree n in t; consequently the integral in (5) or (6) will be elliptic for n=3, 4 and hyperelliptic for $n \geq 5$.

For the elliptic case, the distribution could be computed by proper modification of the aforementioned integrals, so that tables of elliptic functions could be used. (Cf. Hurwicz [7], p. 376)

The following geometrical approach, along the lines of §3 is instructive, and may suggest an extension of the method to values of $n > 3$. For $n = 3$, the problem reduces to finding the mass of the cone

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 0 \qquad (25)$$

where $\lambda_i = \lambda_i(z)$ are determined as the roots of (18). In this case the probability density is

$$\left(\frac{1}{\sqrt{2\pi}}\right)^3 e^{-1/2(x_1^2 + x_2^2 + x_3^2)} \qquad (26)$$

For simplicity, suppose (25) may be written as

$$x_3^2 = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2}\,. \qquad (27)$$ [4]

where $a \geq b$.

By regarding the mass as the limiting sum of elliptical slabs perpendicular to the axis of the cone, the integration may be done in two stages. Consider first

$$\iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} e^{-1/2(x_1^2 + x_2^2)} dx_1\, dx_2 \,. \qquad (28)$$

Setting

$$\begin{aligned} x_1 &= a\, r \cos\theta \\ x_2 &= b\, r \sin\theta \end{aligned} \qquad (29)$$

the integral becomes

$$a\, b \int_{r=0}^{1} \int_{\theta=0}^{2\pi} e^{-1/2r^2(a^2\cos^2\theta + b^2\sin^2\theta)} r.dr\, d\theta \qquad (30)$$

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4. For values of $z$ in the range of (2), there are always two $\lambda$'s of the same sign, and one of opposite sign.

$$= a\,b \int_{r=0}^{1} \int_{\theta=0}^{2\pi} e^{-\frac{1}{2} r^2(b^2 + c^2\cos^2\theta)}\, r \cdot dr\, d\theta \tag{31}$$

where

$$c^2 = a^2 - b^2. \tag{32}$$

Expansion of the integrand gives

$$e^{-\frac{1}{2} r^2 b^2} r \cdot \left[1 - \frac{r^2c^2\cos^2\theta}{2} + \frac{r^4c^4\cos^4\theta}{2^2 \cdot 2!} + \ldots + \frac{(-)^n r^{2n} c^{2n} \cos^{2n}\theta}{2^n \cdot n!} + \ldots\right] \tag{33}$$

which is uniformly convergent for $\begin{matrix} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{matrix}$, hence term-by-term integration is permissible. But

$$\cos^{2n}\theta = \frac{1}{2^{2n-1}}\left[\sum_{k=0}^{n-1} \binom{2n}{k} \cos 2(n-k)\theta + \frac{1}{2}\binom{2n}{n}\right]. \tag{34}$$

(Smithsonian Mathematical Formulae [13].)

Therefore $$\int_0^{2\pi} \cos^{2n}\theta\, d\theta = \frac{1}{2^n} \cdot \frac{(2n)!}{n!n!} \cdot 2\pi \tag{35}$$

and

$$\int_0^{2\pi} e^{-\frac{1}{2} r^2(b^2 + c^2\cos^2\theta)}\, r\, dr\, d\theta = e^{-\frac{b^2r^2}{2}} \cdot 2\pi \cdot r \cdot \left[1 - \frac{r^2c^2}{2} \cdot + \ldots + (-)^n \cdot \frac{r^{2n}c^{2n}}{2^n \cdot n!} \cdot \frac{(2n)!}{2^n \cdot (n!)^2} + \ldots\right] \tag{36}$$

By Stirling's formula (Cramer [14])

$$\frac{(2n)!}{(n!)^3} \sim \left(\frac{2n}{e}\right)^{2n} \sqrt{4\pi n} \cdot \left[\left(\frac{e}{n}\right)^n \cdot \frac{1}{\sqrt{2\pi n}}\right]^3 = \frac{2^{2n} e^n}{n^n} \cdot \frac{1}{n\pi\sqrt{2}} \tag{37}$$

$$= \left(\frac{4e}{n}\right)^n \cdot \frac{1}{n\pi\sqrt{2}}.$$

Consequently the remainder term of the series (36) is of the order

$$\frac{1}{n}\left(\frac{r^2c^2\,4e}{4n}\right)^n \tag{38}$$

which approaches zero rapidly as $n \rightarrow \infty$. Therefore the series in (36) may be approximated within arbitrary accuracy by the first k terms, say. Let

$$\zeta(b, c) = 2\pi ab \cdot \int_0^1 e^{-\frac{1}{2} b^2 r^2} r \left[1 - \frac{r^2c^2}{2} + \ldots + (-)^k \frac{r^{2k}c^{2k}}{2^k \cdot k!} \cdot \frac{(2k)!}{2^k \cdot (k!)^2}\right] dr$$

This can easily be integrated in finite terms. Without presenting further details, it follows that the required mass is

$$2\pi \cdot \left(\frac{1}{\sqrt{2\pi}}\right)^3 \int_{-\infty}^{\infty} a \cdot b \cdot \Gamma(b, c)\, e^{-\frac{1}{2} x_3^2}\, dx_3 \tag{39}$$

where

$$\begin{aligned} a &= k_1 x_3 \\ b &= k_2 x_3 \\ c^2 &= x_3^2(k_1^2 - k_2^2). \end{aligned} \tag{40}$$

$k_1, k_2, k_3$ are constants

The above integral involves only moments of the normal distribution, so the final result is a function of z, which, within an arbitrary degree of accuracy, approximates the mass of the cone in the sample space.

5. <u>The moments of a ratio.</u>

Suppose $X = (X_1, \ldots, X_n)$ has the distribution function $F(x)$ and consider the ratio $\frac{U}{V}$, where $U$, $V$ are functions of $X_1, \ldots, X_n$, and $V \geq 0$. The following device[5] has been used in special instances by Williams [10], Dixon [15], and Hurwicz [7], for finding the moment

$$\mu_k = E\left(\frac{U}{V}\right)^k = \int_{-\infty}^{\infty} \left(\frac{u}{v}\right)^k d\, F(x) \,. \tag{41}$$

Let

$$\psi(t_1, t_2) = \int_{-\infty}^{\infty} e^{t_1 u + t_2 v}\, d\, F(x) \tag{42}$$

be the moment-generating function of U, V. Then $\mu_k$ is presumably given by

$$\mu_k = \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \left[\frac{\partial^k \psi}{\partial t_1^{\,k}}\right]_{t_1 = 0} dt_{21}\; dt_{22} \cdots dt_{2k} \tag{43}$$

where

$$t_2 = t_{21} + t_{22} + \cdots + t_{2k}. \tag{44}$$

Now, since

$$\int_{-\infty}^{0} e^{t\,v}\, dt = \frac{1}{v} \tag{45}$$

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5. Although Williams' article appears to be the first place where such a method has been used, Hurwicz informs me that Williams mentioned to him that Wilks has used it in his lectures on statistics.

if $v > 0$, the following repeated integral will exist and be equal to $\mu_k$ if the $k^{th}$ moment exists.

$$\mu_k = \int_{-\infty}^{\infty} u^k \, d\,F(x) \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} e^{t_2 v} \, dt_{21}\, dt_{22} \ldots dt_{2k}. \tag{46}$$

By Fubini's theorem on Lebesque integration (Saks [16]) the order of integration may be reversed, in which case (46) is the same as (43), with the proviso that differentiation is valid inside the integral sign of (42). Not only must the differentiation referred to be valid, but of course, the partial derivative itself must exist over the domain of integration, namely $(-\infty < t_2 \leq 0)$.

For Q, P, $\Sigma$ defined respectively by (10), (11), (12), Hurwicz [7] has applied (43) to obtain, for $n = 3$, $\mu_1 = \frac{\alpha}{2}\left(1 + \frac{1 - \sqrt{1 - \alpha^2}}{\alpha^2}\right)$. It can be shown that this application of (43) is valid here, if it is possible to find a thin strip which includes the $t_2$ axis from $-\infty$ to 0. The moment generating function is

$$\psi(t_1, t_2) = \phi(-it_1, -it_2) \tag{47}$$

where $\phi(t_1, t_2)$ is given for this case by (23). Now $\psi(t_1, t_2)$ will exist for such $t_1$, $t_2$ that render the quadratic form $x[C^{(n)}(-it_1, -it_2)]x'$ positive definite. It suffices to examine the principal minors of the matrix $C^{(n)}(-it_1, -it_2)$, and find a set of values for which they are all positive. The following set of inequalities must be satisfied simultaneously:

$$y > 0;\ z > 0;\ y\,z - f > 0;\ z^2 - f > 0;\ z - f > 0;\ \det \begin{Vmatrix} y & f & 0 \\ 1 & z & f \\ 0 & 1 & z \end{Vmatrix} > 0 \tag{48}$$

where y, z, f are defined by (24). Let

$$\delta = 1 - 2t_2 . \tag{49}$$

The conditions (48) may be written

$$\delta > 0;\ \delta + \alpha^2 > 0;\ \delta(\delta + \alpha^2) - (t_1 + \alpha)^2 > 0;\ (\delta + \alpha^2)^2 - (t_1 + \alpha)^2 > 0;$$

$$\delta + \alpha^2 - (t_1 + \alpha)^2 > 0;\ \delta[(\delta + \alpha^2)^2 - (t_1 + \alpha)^2] - (t_1 + \alpha)^2(\delta + \alpha^2) > 0. \tag{50}$$

On setting $t_1 = 0$ in all these inequalities, it is seen they hold simultaneously for $\delta > 1$. Hence, by continuity of $C^{(n)}(-it_1, -it_2)$, a sufficiently thin strip can be found which includes the $t_2$ axis from $-\infty$ to 0, such that all the principal minors are positive. The form of $\psi(t_1, t_2)$, given by (47) and (23) shows that all orders of the partial derivative with respect to $t_1$ exist in this strip; further, differentiation inside the integral sign of (42) is also permitted. Consequently the procedure used above, by Hurwicz, is valid, not only for $n = 3$, but for all values of n, and for all moments.

The author has applied the same technique to find the second moment of the above statistic. The result yields integrals which are not finite, but it remains to be examined carefully before it can be concluded that the second moment is infinite. This conclusion seems plausible, however, in view of the fact that the $X P X'$ has finite probability density at zero for the case $n = 3$, with $\Sigma = I_n$ and P defined by (11).

6. Expansion of distribution by Gram-Charlier series.

Let X be a random variable with distribution F(x), and set

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \tag{51}$$

and

$$\phi^{(\nu)}(x) = \left(\frac{d}{dx}\right)^{\nu+1} \Phi(x) \qquad (\nu = 0, 1, 2, \ldots) . \tag{52}$$

Let

$$c_\nu = (-1)^\nu \int_{-\infty}^{\infty} H_\nu(x)\, d\,F(x) \tag{53}$$

where $H_\nu(x)$ is the $\nu^{th}$ Hermite polynomial defined by

$$\left(\frac{d}{dx}\right)^{\nu} e^{-\frac{x^2}{2}} = (-1)^\nu H_\nu(x)\, e^{-\frac{x^2}{2}} . \tag{54}$$

Cramer [14] states that whenever the integral

$$\int_{-\infty}^{\infty} e^{\frac{x^2}{4}} d\,F(x) \tag{55}$$

is convergent, we may write

$$F(x) = \sum_{\nu=0}^{n} \frac{c_\nu}{\nu!} \phi^{(\nu)}(x) + R_n(x) \tag{56}$$

where

$$\lim_{n\to\infty} R_n(x) = 0 \quad \text{for all } x. \tag{57}$$

If the method of § 3 is applied in conjunction with such a series as (56), the expansion is greatly simplified, but its validity must be established. The constants $c_\nu$, in this case, may be found from the characteristic function (22), as functions of z, and assuming convergence, the following series represents the distribution function G(x) of (4).

$$G(x) = H_z(0) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot c_{2n}^{(z)}}{2^n \cdot n!} \tag{58}$$

since

$$H_{2\nu}(0) = (-1)^\nu \cdot 1.3.5. \dots (2\nu - 1).$$
$$H_{2\nu+1}(0) = 0. \tag{59}$$

The validity of this expansion is now under investigation.

It may be remarked here that Robbins [5], [6], in considering the problem of finding the distribution of $x\,Q\,x'$, and of $\frac{x\,Q\,x'}{x\,P\,x'}$ where Q and P are diagonal matrices with positive elements, and $\Sigma = I$, gives results in series expansions. An asymptotic expansion exhibiting the approach to normality as the sample size increases, for three specific ratios (2) occurring in serial correlation, is given for the case $\Sigma = I_n$, by Hsu [17].

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