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Estimation of a System of Equations with Restricted Σ

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Let us estimate $A_{u_I x}$ and $\Sigma_{I,I}$ where

(1) $A_{u_I x} X_t' = u_{tI}'$,

(2) $A_{u_{II} x} X_t' = u_{tII}'$,

(3) $\mathcal{E}(u_{ti}' u_{tj}) = \Sigma_{i,j}$, $\qquad i,j = I, II$,

(4) $u_t = (u_{tI}, u_{tII})$,

(5) $\mathcal{E}(u_s' u_t) = 0$, $\qquad s \neq t$

(6) $x = (y,z)$,

(7) u_t are distributed normally with mean 0,

(8) there are no restrictions on $A_{u_{II} x}$, $\Sigma_{I,II}$, and $\Sigma_{II,II}$.

Then we may replace $A_{u_{II} x}$ by $A_{u_{II} x} - \Sigma_{II,I} (\Sigma_{I,I})^{-1} A_{u_I x}$ obtaining in the new model $\Sigma_{I,II} = 0$.

Then the logarithm of the likelihood function becomes

(9) $$\frac{1}{T} \log L = -\frac{G}{2} \log 2\pi + \frac{1}{2} \log \left| A_{uy} \right|^2 + \frac{1}{2} \log \left| \Sigma_{I,I}^{-1} \right| + \frac{1}{2} \log \left| \Sigma_{II,II}^{-1} \right| - \frac{1}{2} \operatorname{tr} \Sigma_{I,I}^{-1} A_{u_I x} M_{xx} A'_{u_I x} - \frac{1}{2} \operatorname{tr} \Sigma_{II,II}^{-1} A_{u_{II} x} M_{xx} A'_{u_{II} x}$$

If we maximize log L with respect to $\Sigma_{II,II}$ which is unrestricted, we obtain $\Sigma_{II,II} = A_{u_{II} x} M_{xx} A'_{u_{II} x}$. Substituting in (9), we obtain

(10) $$\frac{1}{T} \log \hat{L} = -\frac{G}{2} \log 2\pi - \frac{G_{II}}{2} + \frac{1}{2} \log \left| A_{uy} W_{yy} A'_{uy} \right| - \frac{1}{2} \log \left| W_{yy} \right| + \frac{1}{2} \log \Sigma_{I,I}^{-1} - \frac{1}{2} \operatorname{tr} \Sigma_{I,I}^{-1} A_{u_I x} M_{xx} A'_{u_I x} - \frac{1}{2} \log \left| A_{u_{II} x} M_{xx} A'_{u_{II} x} \right|.$$

Let us now maximize $A_{u_{II} x}$ given $A_{u_I x}$. We may rewrite (10) as

(11) $$\frac{1}{T} \log \hat{L} = f(A_{u_I x}, \Sigma_{I,I}) + \frac{1}{2} \log \left| A_{u_{II} y} W_{yy \cdot u_I} A'_{u_I y} \right| - \frac{1}{2} \log \left| A_{u_{II} x} M_{xx} A'_{u_{II} x} \right|$$

(12) $f(A_{u_I x}, \Sigma_{I,I}) = -\frac{G}{2} \log 2\pi - \frac{G_{II}}{2} - \frac{1}{2} \log \left| W_{yy} \right| +$
$\frac{1}{2} \log \left| A_{u_I y} W_{yy} A'_{u_I y} \right| + \frac{1}{2} \log \left| \Sigma^{-1}_{I,I} \right| - \frac{1}{2} \operatorname{tr} \Sigma^{-1}_{I,I} A_{u_I x} M_{xx} A'_{u_I x}$

If we maximize (11) with respect to $A_{u_{II} z}$, we obtain, as in my thesis, $A_{u_{II} z} = -A_{u_{II} y} \Pi_{yz}$. Let us substitute this in (11). We obtain to be maximized.

(13) $F = \dfrac{\left| A_{u_{II} y} W_{yy \cdot u_I} A'_{u_{II} y} \right|}{\left| A_{u_{II} y} W_{yy} A'_{u_{II} y} \right|}$

It is apparent that $F \leq 1$. However, we can take $A_{u_{II} y} W_{yy} A'_{u_I y} = 0$, since $A_{u_{II} y}$ is unrestricted. Then $F = 1$, and hence

(14) $\log \hat{L} = f \; (A_{u_I x}, \; \Sigma_{I,I})$.

If $\Sigma_{I,I}$ is unrestricted, my result (3.83) follows just as (10) was derived from (9).

Note: The paper of mine referred to by Koopmans in Cowles Commission Discussion Papers: Statistics 302 is my thesis.