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REMARKS ON A RATIONAL SELECTION OF A DECISION FUNCTION

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Problems in the theory of statistics seem to reduce to the selection of a strategy from several available strategies. The problem of the selection of such a strategy is analogous to the selection of a strategy in the two person zero sum game where nature is the statistician's opponent. One difference between the two problems lies in the fact that it does not seem proper to regard the unknown state of nature as a strategy of an opponent who wishes to maximize your loss.

In the case where there are a finite number of pure strategies d_1, d_2, ..., d_m available to the statistician and a finite number of conceivable states of nature S_1, S_2, S_n, there exists a finite risk matrix $R = \|r_{ij}\|$ r_{ij} = expectation of the loss incurred by statistician if he selects strategy d_i when S_j is the state of nature. The development and use of the Risk Matrix depend on the existence of a utility which has the property that if a statistician has probability p of reaching at a certain time a state in which he has utility u_1 and $q = 1-p$ of reaching a state at that time in which he has utility u_2, then his utility for that time is $p\,u_1 + q\,u_2$.

Clearly, if there exists a d_i so that $r_{ij} \leq r_{kj}$ for all k and j, then d_i is at least as good as all the other strategies and if also for every k there is some j so that $r_{ij} < r_{kj}$ then d_i is uniformly better than the other strategies. Unfortunately it is seldom possible to find such a strategy. The problem remains of phrasing a reasonable criterion for selecting a strategy in the case where uniformly best strategy fails to exist.

Wald has mentioned that a conceivable criterion would be to minimize the maximum risk. That is to choose a strategy d_i if $\max_j r_{ij} \leq \max_j r_{kj}$ for all k.

This is the criterion which is used in the Theory of Games. This criterion is equivalent to ranking strategies in preference according to $-\max r_{ij}$ i.e. if $\max_j r_{ij} \leqq \max_j r_{kj}$ d_i is $\begin{matrix}\text{better than}\\ \text{as good as}\end{matrix}$ d_k.

As in the Theory of Games we may introduce mixed strategies i.e. strategies which consist of selecting a pure strategy d_i at random where the probability of selecting d_i is ξ_i and $\sum_{i=1}^{m} \xi_i = 1$ $\xi_i \geqslant 0$. A result in the Theory of Games and Economic Behavior by Von Neumann and Morgenstern is that $v = \min_{\xi} \max_{\eta} \sum \xi_i \eta_j r_{ij} = \max_{\eta} \min_{\xi} \sum \xi_i \eta_j r_{ij}$. This result seems to imply that in the zero sum two person game the mixed strategies corresponding to v are rational strategies on the part of the opponents.

Some criticism was leveled at the criterion of minimizing the maximum risk for the following reason. It is a criterion which is useful if "nature" is the statisticians opponent, but one can hardly assume this. If the statistician applies this criterion in the case where one of the states is a state of disaster then one finds oneself forced to decide by just considering the risks in the case of the state of disaster. Consider the following example of a risk matrix:

	S_1	S_2	$\max_j r_{ij}$
d_1	1	100	100
d_2	99	99	99

The criterion of minimizing the maximum risk prefers d_2 to d_1

If "nature" is malevolent it will select strategy S_2 and our statistician is wise to select strategy d_2. If one does not assume that nature is malevolent it seems unreasonable to allow the slight preference of d_2 to d_1 is the case of disaster to dictate the choice of d_2. Indeed if our statistician were to follow the policy of minimizing the maximum risk, he might very well commit suicide rather than risk being unpleasantly killed in an accident. Savage felt that a better criterion would be to minimize the maximum regret.[1] Consider $\min_i r_{ij}$. This represents the minimum risk that can be incurred by the statistician if state j is the actual state of nature. If the statistician select d_i and state j turns out to be the actual state one may say that his regret for choosing strategy i would be $r_{ij} - \min_i r_{ij} = \rho_{ij}$. (Here there is an assumption that the "regret" of attaining a state of utility a instead of one of utility b is measured by b-a.) The Savage criterion of minimizing the maximum regret would be to rank the available strategies in preference according to $-\max_j \rho_{ij}$. i.e. if $\max_j \rho_{ij} \overset{=}{<} \max \rho_{kj}$, d_i is as good as / better than d_k. In the previous example:

R	S_1	S_2	ρ	S_1	S_2
d_1	1	100	d_1	0	1
d_2	99	99	d_2	98	0

The criterion of minimizing the maximum regret prefers d_1 to d_2

While the method of minimizing the maximum regret seems to overcome the difficulties facing the method of minimizing the maximum regret, it too seems to have some disadvantageous properties. Consider the risk matrix:

R	S_1	S_2	S_3
d_1	10	0	14
d_2	0	8	15
d_3	80	80	0

If d_3 were not an available decision the matrix would be,

R			
d_1	10	0	14
d_2	0	8	15

(1) The criterion of minimizing the maximum regret was transmitted to me via K. J. Arrow.

In the first case d_1 is the strategy that would be selected by the principle of min max regret. (Let us ignore mixed strategies for the time being.) In the second case d_2 is the one which would be selected. Thus it happens that when a new strategy is made available, the preferred strategy is neither the one that was the preferred nor the new one, but the one that was formerly not preferred.

Arrow conjectured that this was probably due to some inherent intransitivity on the ranking and with his help the following case was constructed.

	S_1	S_2	S_3
d_1	10	0	14
d_2	0	8	15
d_3	5	7	8

If comparisons where respectively d_1 and d_2, d_2 and d_3, d_3 and d_1 are considered, one sees that d_1 is preferred to d_2 , d_2 to d_3, and d_3 to d_1 if the min max regret is applied.

Some question arose as to whether the first criticism of min max regret was fair in inasmuch as mixed strategies were ignored. The following example was constructed to show that the use of mixed strategies does not avoid the pitfall.

	S_1	S_2	S_3
d_1	10	0	14
d_2	0	8	15
d_3	80	80	0
d_4	12.922	12.922	12.922

If a best (by the min max regret criterion) strategy were to be constructed when d_1, d_2, d_3, d_4 were available it would be equivalent to d_4 in that the risks would be the same. If ,however, one considers only d_1, d_2, d_4 available then obviously d_2 would be better than d_4. Indeed the "best" mixed strategy involving d_1 d_2 d_4 (d_3 unavailable) would be some d_5 which is equivalent to a mixed strategy of d_1 d_2 d_3 d_4 but is not best as long as d_3 is available. From another point of view while d_4 was always available it did not become the "best" until d_3 was thrown into the picture.