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Remarks on a Rational Selection of a Decision Function

At this point it seems advisable to axiomatize the concept of the rational selection of a decision function. There seem to be various methods of phrasing the actual problem in an axiomatic framework. We shall consider one such framework and a set of axioms which seem to be reasonable requirements.

Whether or not a criterion satisfies the axioms is closely related to the set of problems (domain) over which the criterion is supposed to exist. For example it is conceivable that a suitable criterion may be framed for the class of all problems with one or two states of nature (columns) but that none may exist for the class of all problems with three or less states of nature. Another consideration in this direction arises from the question as to whether it makes sense to insist that a criterion apply to the selection from any set of strategies. One may feel that it is not proper to insist on applicability to problems where not all mixed strategies are available.

In the following the solution of a problem will represent a set of the available strategies so that no one of these strategies is "worse" than any available strategies and so that there seems no indication to prefer one strategy of the solution to any others of the solution. This nonpreference is not necessarily indifference.

In what follows considerable space is devoted to framing relatively simple concepts such as the invariance of the solution of a problem if several columns are interchanged. The complications arise in part out of an attempt to give the notation enough generality to be applicable to problems where an infinity of states of nature and of strategies are present. This notation is needed because it is impossible to avoid the infinite cases even in simple problems as the existence of mixed strategies makes evident. Another source of complications is the need to compare two problems where one is the same as the other except for the interchange of several rows.

At this point a change in notation will be made. Instead of a risk matrix, we shall use the payoff matrix, the elements of which are the payoffs in utility.[1]

DEFINITION 1: A problem Q is a real valued function (with $+\infty$ and $-\infty$ as possible values) defined on the Cartesian product of two non-null spaces D and S. The value of the function is called the payoff $u = u(d, s)$; $d \in D$, $s \in S$. D = the set of strategies, S = the set of states of nature. We may write $Q = (u, D, S)$.

DEFINITION 2: A general Problem G is a class of problems Q.

DEFINITION 3: In a problem Q, to an element d of D, there corresponds the function on S, $u_d = u(d, s)$, $s \in S$. This function is the "d-row" of the problem. Q. We may write "d-row of Q" $= d_q = (u_d, S) = (u, \{d\}, S)$.

DEFINITION 4: Similarly the "s-column of Q" $= s^q = (u^s, D) = (u, D, \{s\})$ where $u^s = u(d, s)$, $d \in D$.

It is evident the problem Q is uniquely determined by its d-rows or by its s-columns.

DEFINITION 5: ξ_D is a probability defined on D and is called a mixed strategy of D. D_* will represent a set of mixed strategies of D. When there is no ambiguity D may be omitted from ξ_D.[2]

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1. From a historical point of view risk was originally phrased as the negative of the payoff, but it was actually used and thought of as a regret. Thus the two terms risk and regret are ambiguous and will be dropped.

2. We may look upon a mixed strategy on D_* as a mixed strategy on D since their interpretations make them identical.

DEFINITION 6: η^S is a probability defined on S and is called a mixed state of S. S^* will represent a set of mixed states of S. When there is no ambiguity S may be omitted from η^S.

DEFINITION 7: Corresponding to $Q = (u, D, S)$, D_* and S^*_* (u, D_*, S^*) is the problem where the new u is an extension of the old one.

$u = u(\xi_D, \eta^S) = E\{u(d, s)\}$ with respect to the probability $\xi_D \otimes \eta^S$ on $(D \otimes S)$. This is the extension of Q with respect to S^*, and D_*. (E = Expectation)

It should be noted that $E\{u(d, s)\}$ may be undefined depending on the original u, ξ_D, and η^S. In general we shall have D_* and S^* such that $u(\xi_D, \eta^S)$ is defined (and possibly $+\infty$ or $-\infty$).

DEFINITION 8: If D_* is the set of all mixed strategies ξ_D of D so that $u(\xi_D, S)$ is defined, (u, D_*, S) is the mixed problem of (u, D, S).

DEFINITION 9: $Q_1 \textcircled{D} Q_2$ (read Q_1 is strategically contained in Q_2) if $D_1 \subset D_2$, $S_1 = S_2$ and for $d \in D_1$, $u_1(d, s) = u_2(d, s)$.

DEFINITION 10: Q_1 d b d r r/Q_2 (read Q is derived by deleting repetitious rows of Q_2) if $Q_1 \textcircled{D} Q_2$ and for d_2 in D_2, there is a d_1 in D_1 so that $u_1(d_1, s) = u_2(d_2, s)$.

DEFINITION 11: Q_1 d b d r c Q_2 (read Q_1 is derived by deleting repetitious columns of Q_2) if $S_1 \subset S_2$, $D_1 = D_2$ and for $s \in S_1$ and S_2, $s^{q_1} = s^{q_2}$ and for s_2 in S_2 but not in S_1, there is an s_1 in S_1 so that $s_1^{q_1} = s_2^{q_2}$.

DEFINITION 12: $Q_1 \overset{D}{=} Q_2$ (read Q_1 is D equivalent to Q_2) if there is a one to one transformation f of D_1 onto D_2 so that $u_1(d_1, s) = u_2(f(d_1), s)$.

DEFINITION 13: $Q_1 \overset{S}{=} Q_2$ (read Q_1 is S equivalent to Q_2) if there is a one to one transformation g of S_1 onto S_2 so that $u_1(d, s_1) = u_2(d, g(s_1))$.

DEFINITION 14: Q_1 is isomorphic to Q_2 if there is a one to one transformation f of D_1 onto D_2 and a one to one transformation g of S_1 onto S_2 so that $u_1(d_1, s_1) = u_2(f(d_1), g(s_1))$.

DEFINITION 15: If $Q_1 = (u_1, D_1, S)$, $Q_2 = (u_2, D_2, S)$, and $u_1(d, s) = u_2(d, s) = u(d, s)$, if $d \in D_1 \cap D_2$ then $Q_1 \circ Q_2 = (u, D_1 \cap D_2, S)$, note that $Q_1 \circ Q_2 ⓘ Q_1$.

DEFINITION 16: If $Q_1 = (u_1, D_1, S)$, $Q_2 = (u_2, D_2, S)$, and $u_1(d, s) = u_2(d, s) = u(d, s)$ if $d \in D_1 \cap D_2$ then $Q_1 + Q_2 = (u, D_1 \cup D_2, S)$ where $u(d, s) = u_1(d, s)$ if $d \in D_1$, $u(d, s) = u_2(d, s)$ if $d \in D_2$. Note that $Q_1 ⓘ Q_1 + Q_2$. Further if $Q_1 ⓘ Q_2$, there is a Q_3 so that $Q_1 + Q_3 = Q_2$.

DEFINITION 17: $Q_1 \otimes Q_2$ is a problem $Q = (r, D, S)$ where $D = \left\{(d_1, d_2) \mid d_1 \in D_1, d_2 \in D_2\right\}$, $S = \left\{(s_1, s_2) / s_1 \in S_1, s_2 \in S_2\right\}$, and $u((d_1, d_2), (s_1, s_2)) = u_1(d_1, s_1) + u_2(d_2, s_2)$.

The above notation mean that D is the set of ordered pairs such that the first element is in D_1 and the second in D_2. Essentially, $Q_1 \otimes Q_2$ is the problem of playing Q_1 and then Q_2.

DEFINITION 18: A solution C of a problem Q is a subset of D.

DEFINITION 19: A solution of a general problem G is a function C(Q) defined on G so that for each Q in G, C(Q) is a solution of Q.

DEFINITION 20: In a problem Q, the d_1-row d_{1q} is uniformly better than the d_2-row d_{2q} if $u(d_1, s) \geq u(d_2, s)$ for all $s \in S$ and $u(d_1, s) > u(d_2, s)$ for some $s \in S$. d_{1q} is uniformly equal to d_{2q} $u(d_1, s) = u(d_2, s)$ for all $s \in S$.
Using these relations we may define a partial ordering on the set of d - rows of a problem Q, we write d_{1q} P d_{2q} (preference) if d_1 is uniformly better than d_2

d_{1q} I d_{2q} (indifference) if d_1 is uniformly equal to d_2

d_{1q} R d_{2q} if $d_1 P d_2$ or $d_1 I d_2$

d_{1q} N d_{2q} (non comparability) if neither $d_{1q} R d_{2q}$ nor $d_{2q} R d_{1q}$

Thus $d_{1q} N d_{2q}$ if there are s_1 and $s_2 \in S$ so that $u(d_1, s_1) > u(d_2, s_1)$ and $u(d_1, s_2) < u(d_2, s_2)$. It should be noted that the partial ordering may be undefined if some risks are $+\infty$ or $-\infty$ or are undefined. In general we shall avoid this case although it may sometimes be treated. In place of the above notation we shall use d_1Pd_2, etc., whenever there is no ambiguity, always remembering that the partial ordering is always with reference to a particular problem.

Now that a number of formal definitions have been made to facilitate future discussion we are faced with defining a rational solution of a general problem. Such a definition will involve satisfying axioms of rationality. To frame such axioms we must ask what is intuitively meant by a solution and what role does G play.

First of all a solution in the following should be considered as a set of strategies which remain after throwing away "bad" strategies and eliminating some which while not necessarily bad, still fail to make the grade. A rational statistician must then make a choice of one of strategies in the solution. If we were to consider any one of these to be as "good" as any other, then it would make no difference which he chose. However it is the author's intention to assume that they are not necessarily as good as one another, that one element of the solution may be replaced by a uniformly better strategy without evicting the other element from the solution. It seems reasonable that if two strategies are in the solution, any mixed strategy of the two should also be in the solution. This will be incorporated into the axioms but it is felt that the consequences of omitting this axiom will prove useful in later work. These consequences will be considered.

Since a solution is phrased with respect to a class G, the question arises of what to do if $Q \in G_1$, $Q \in G_2$, and C(Q) with respect to G_1 has no strategies in common with C(Q) with respect to G_2. That the use of general problems is not altogether meaningless follows from the fact that for every $Q \in G$, D should contain its mixed strategies. It must be admitted that another reason for the use of G stems from the author's feeling that if G included all problems where D contains all of its mixed strategies no solution would be found. Then it would be nice if a justification could be made for the existence of a meaningful set of G's which are disjunct. The only such justification that the author can attempt (though with some misgivings) is the following. In any problem that is to be faced there is a certain specific set of states of nature. Thus the set of all Q's may be put into G's where Q_1 and Q_2 are in a G if and only if there is a one to one correspondence between S_1 and S_2. (Each G corresponds to a distinct cardinal number.)

Another justification for the use of G's which however fails to guard against a Q being in two G's, is a claim that is often made that in certain classes of problems, each row is unimodal. That is, there is a complete ordering of S, so that for each row there is an s_d, so that u(d, s) is monotone increasing for $s < s_d$ and monotone decreasing for $s > s_d$.

DEFINITION 21: A general problem G is rationally solvable by C(Q) if the following axioms are satisfied.

This definition is quite pretentious in that it is conceivable that C(Q) may satisfy the following axioms but fail to satisfy a relevant axiom which has been forgotten but which should be satisfied for one to claim the solution to be rational.

AXIOM 1: C(Q) is non-null for every Q in G.

AXIOM 2: If $d_1 I d_2$, $d_1 \in C(Q)$, $Q \in G$, then $d_2 \in C(Q)$.

AXIOM 3: If $d_1 \in C(Q)$, $Q \in G$, then there is no $d_2 \in D$ so that $d_2 P d_1$.

AXIOM 4: If Q_1 [illegible] Q_2, Q_1, $Q_2 \in G$, then $C(Q_1) = D_1 \cap C(Q_2)$. This means that all strategies of the solution which are available after deletion are in the solution and none others.

AXIOM 5: If $Q_1 \textcircled{D} Q_2$, $Q_1, Q_2 \in G$, $C(Q_2) \subset C(Q_1) \cup (D_2 - D_1)$. This axiom says that if the available strategies of Q_1 are increased, then each element of the solution of the new problem is an element of the solution of the old problem or one of the new strategies. This axiom is equivalent to $C(Q_2) \cap D_1 \subset C(Q_1)$.

AXIOM 6: If $Q = Q_1 + Q_2$, $Q, Q_1 \in G$, where for every $d_2 \in D_2$ there is a $d_1 \in D$, so that $d_1 P d_2$ then $C(Q) = C(Q_1)$.

It is not too unreasonable to consider extending Axiom 6 to Axiom 6a. However it would be wise to avoid making use of the extension.

AXIOM 6a: If $Q = Q_1 + Q_2$, $Q, Q_1, Q_2 \in G$, where $C(Q)$ contains no elements of D_2 then $C(Q_1) = C(Q)$.

This may be interpreted to mean that if from a problem Q strategies not in the solution are deleted, the solution remains unchanged.

AXIOM 7: If $Q = Q_1 + Q_2$, $Q, Q_1, Q_2 \in G$, then $C(Q_1) \cap C(Q_2) \subset C(Q_1 + Q_2)$.

AXIOM 8: If Q_1 d b d r c. Q_2, Q_1, Q_2 G, then $C(Q_1) = C(Q_2)$. In other words if two columns representing states s_1 and s_2 are the same, the solution is is the same as that of the problem where the states s_1, s_2 are replaced by the state s_1 or s_2.

AXIOM 9: If $Q_1, Q_2 \in G$, $Q_1 \overset{D}{=} Q_2$, then $C(Q_2) = f[C(Q_1)]$. This says that the solution does not depend on the names of the strategies but on the payoff's.

AXIOM 10: If $Q_1, Q_2 \in G$, $Q_1 \overset{S}{=} Q_2$, then $C(Q_2) = C(Q_1)$.

Consequence 1. If $Q_1, Q_2 \in G$, Q_1 isomorphic to Q_2, and there is a $Q_3 \in G$ so that $Q_1 \overset{S}{=} Q_3 \overset{D}{=} Q_2$ then $C(Q_2) = f[C(Q_1)]$. This is immediate from axioms 9 and 10.

AXIOM 11: If $Q_1 = (u_1, D, S)$, $Q_2 = (u_2, D, S)$, $Q_1, Q_2 \in G$, $u_2(d, s) = pu_1(d, s) + (1 - p)\ u_0(s)$ $0 < p \leq 1$, and $u_0(s)$ is attainable[1] then $C(Q_2) = C(Q_1)$.

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1. It is conceivable that there is no future for which the utility exceeds a certain number K. Then to insert $u_0(s) = K + 1$ would be meaningless in view of the justification of Axiom 13. The set of attainable utilities is evidently an interval.

This axiom which was just conveyed to me by Herman Rubin, is justified as follows. If one were told that he would be given a problem Q_1 if a coin fell heads and a strategy would be forced on him if the coin fell tails, then his strategy if the coin fell heads would be $C(Q_1)$. The game including the coin has payoff $u_2(d, s) = pu_1(d, s) + (1 - p)\, u_o(s)$ where p is the probability of falling heads (and is not zero) and $u_o(s)$ is the payoff under the strategy forced on him if the coin fell tails. The rationale of this axiom can be considered as a consequence of the unstated axiom of rational behaviors that a rational person will react in the same fashion to the same situation facing him irrespective of what came before. This axiom is rather powerful and has some important consequences.

Before stating these consequences let us define certain mixed strategies.

DEFINITION 22: $(d_1, d_2, \ldots, d_n; p_1, p_2 \ldots, p_n)$ is the mixed strategy consisting of selecting d_i with probability p_i, $i = 1, 2, \ldots, n$ where $p_1 + p_2 + \ldots + p_n = 1$.

Consequence 2. If $Q \in G$, and $Q_1 \overset{D}{=} Q_2 \textcircled{D}\, Q$ implies $Q_1, Q_2 \in G$, then $\xi = (d_1, d_2; p, 1 - p) \in C(Q)$ for $o < p < 1$ implies d_1 and $d_2 \in C(Q)$. (This is a converse of convexity.)

Proof: Let $Q_p = (u_p, D, S)$ where $u_p(d, s) = pu(d, s) + (1 - p)\, u(d_2, s)$

$$Q_{2p} = (u, D_{2p}, S) \text{ where } D_{2p} = \left\{ (d, d_2; p, (1 - p)) \,/\, d \in D \right\}$$

Then it is evident that for $p > 0$, $Q_p \overset{D}{=} Q_{2p}\, D\, Q$

Thus $Q_p, Q_{2p} \in G$. By axiom 5, $\xi\, C(Q_{2p})$ by axiom 9, $d_1 \in C(Q_p)$

Then by axiom 11 applied to Q_p and Q, where $u_o(s) = u(d_2, s)$ we have $d_1 \in C(Q)$.

Similarly $d_2 \in C(Q)$.

This consequence is not only a converse of convexity but it just about states that mixed strategies have no special claim to fame. This is not too surprising in view of the fact that the game theory interpretation of the role of mixed strategies was that they served to prevent the opponent from guessing at one's own strategy. Nature could hardly be expected to be attempting to outguess the statistician.

For future discussions it will be convenient to assume that $u = 0$ is an attainable value and that the interval of attainable u's contains $u = 0$ as interior point. That we may make these assumptions is evident from the fact that if u is a utility indicator derived from the axioms of von Neumann and Morgenstern so is $au + b$, $a > 0$. Furthermore if the set of attainable states were indifferent (only one utility) then our problems would disappear.

Consequence 3. If $Q_1, Q_2 \in G$, $Q_1 = (u_1, D, S)$, $Q_2 = (u_2, D, S)$ where $u_2(d, s) = au_1(d, s)$ $a > 0$, then $C(Q_2) = C(Q_1)$.

We may assume $a \leq 1$ for otherwise we can reverse Q_1 and Q_2. If $a = 1$, the consequence is trivial. If $0 < a < 1$, we let $u_o(s) = 0$ (which is attainable) and apply axiom 11.

Consequence 4. If $Q_1, Q_2, Q_{3p} \in G$, $Q_1 = (u_1, D, S)$, $Q_2 = (u_2, D, S)$, $Q_{3p} = (u_{3p}, D, S)$, $0 < p < 1$, where $u_2(d, s) = u_1(d, s) + u_o(s)$, $u_{3p}(d, s) = pu_2(d, s)$, then $C(Q_2) = C(Q_1)$.

Proof: $u_{3p}(d, s) = pu_1(d, s) + (1 - p)\left[\frac{p}{1 - p} u_o(s)\right]$

For p small enough $\frac{p}{1 - p} u_o(s)$ is an attainable utility since $|u_o(s)| \leq |u_2(d, s)| + |u_1(d, s)|$. Then $C(Q_3) = C(Q_1)$ by axiom 11. But by consequence 3, $C(Q_3) = C(Q_2)$.

Finally we have the convexity axiom.

AXIOM 12: If $Q \in G$, and ζ is a mixed strategy of the elements of $C(Q)$ [i.e. ζ is a probability on D such that the probability of $C(Q)$ is one] then $\zeta \in D$ implies $\zeta \in C(Q)$.

As was mentioned before we shall avoid using the implications of axiom twelve until we have exhausted the other axioms. It may be noted that for any G, the solution C(Q) equal the set of all admissible strategies of Q [those for which there are no uniformly better ones] constitutes a solution satisfying axioms two to eleven but not twelve and not one. By restricting G so that every $Q \in G$ contains admissible strategies, axiom one will be satisfied. That this axiom system is consistent follows from considering G the set of Q's with one column and attainable payoffs and the least upper bound payoff of each problem attained in that problem. Of course axiom 8 is trivially satisfied in this case.

Let us now consider the class G of all problems Q containing two columns and such that Q is the mixed problem of a problem with a finite number of rows [every payoff must of course be attainable]. Thus every mixed strategy of D is in D. We denote the interval of attainable utilities by (A, B). This interval may be open, half open, closed, finite, semi-infinite or infinite.

A problem Q in G is S equivalent to a problem in which one of the states of nature is I and the other II. [These are the labels put on the states of nature.] Thus it may be geometrically represented by a set of points in the (x_1, x_2) plane where $x_1 = u(d, I)$, $x_2 = M(d, II)$.

Since D contains all mixed strategies we have for each x, y, p, x, y in the set, $0 \leq p \leq 1$, the point $px + (1 - p)y$ in the set. The original pure strategies correspond to a finite number of points and hence the mixed problem consists of the smallest convex set containing those points. This set is of course a closed polygon and every vertex corresponds to a pure strategy. Two different strategies may correspond to the same point (if they are indifferent). This convex set associated with the problem Q is called a geometrical representation of Q. The questions arise as to how many representations may a given problem have and of what problems is a given set the geometrical representation. First of all a given problem may have (at the most) two representations, for once the states are labeled, the problem determines the representation. There are two possible labelings and thus at most two representations. To answer the other question we first construct a problem which is uniquely determined by a geometrical representation. $\tilde{Q} = (\tilde{u}_1, \tilde{D}, \tilde{S})$ where $\tilde{S} = \{I, II\}$, $\tilde{D} = \{(x_1, x_2) \mid (x_1, x_2)$ is in the convex set of the representation$\}$ $\tilde{u}((x_1, x_2), I) = x_1$, $\tilde{u}((x_1, x_2), II) = x_2$. There is no ambiguity in identifying this problem $\tilde{Q}$ with the set of points described above and calling both a geometrical representation of Q for there is a one to one correspondence between the sets and the $\tilde{Q}$. [It may be noted that while $\tilde{Q}$ is not a mixed problem $\tilde{Q}$ d b d r r from a mixed problem for this G], it is evident that there is a two column problem Q_1 with attainable payoffs so that Q_1 d b d rr Q and $\tilde{Q}$ is isomorphic to Q_1. Also if there is a two column problem Q_1 with attainable payoffs so that Q_1 d b d rr Q and $\tilde{Q}$ is isomorphic to Q_1 then $\tilde{Q}$ is a geometrical representation of Q.

Under what conditions do two problems Q_1 and Q_2 in G have the same geometrical representation. This may occur if and only if there are two problems Q_3 and Q_4 in G so that Q_1 d b d rr Q_3, Q_2 d b d rr Q_4, Q_3 is isomorphic to Q_4. Furthermore Q_3 and Q_4 have the same representations as Q_1 and Q_2 and a row of Q_3 corresponds to the same point [i.e. row of $\tilde{Q}$] as does the corresponding row of Q_4 in the isomorphism if the states are properly labelled. By axiom 2 if a row of Q_3 is in $C(Q_3)$, every uniformly equal strategy is in $C(Q_3)$ and thus every row of Q_3 corresponding to a point in $\tilde{Q}$ which corresponds to a strategy of $C(Q_3)$ is in $C(Q_3)$. Thus $C(Q_3)$ uniquely determines a subset of $\tilde{Q}$ which in turn uniquely determines $C(Q_3)$. By consequence 1, $C(Q_4) = f(C(Q_3))$. The nature of this isomorphism is that a strategy of Q_4 corresponds to a strategy of Q_3 only if they both correspond to the same point.

Thus $C(Q_3)$ and $C(Q_4)$ determine the same subset of $\tilde{Q}$. But by axiom 4 this subset is exactly the subset determined by $C(Q_1)$ and $C(Q_2)$. Thus this subset corresponds to all problems which have the representation $\tilde{Q}$. Let us then call this subset $C(\tilde{Q})$ even though $\tilde{Q}$ is technically not in G. It is well to note here that consequence 1 implies that if $\tilde{Q}$ is reflected about the line $x_1 = x_2$, $C(\tilde{Q})$ is similarly reflected. We may also note the partial ordering is maintained in going from two rows in Q to the corresponding ones in $\tilde{Q}$ and vice versa.

Thus far we have been less specific than we may have been in the hope of extending these concepts. However it is well to note that for the particular G we are discussing $\tilde{Q}$ is a closed convex polygon with a finite number of vertices (each representing at least one pure strategy) and this polygon lies in the rectangle where (x_1, x_2) are both in the interval (A, B). Conversely for every such $\tilde{Q}$ there are problems of G which have this $\tilde{Q}$ as a geometrical representation.

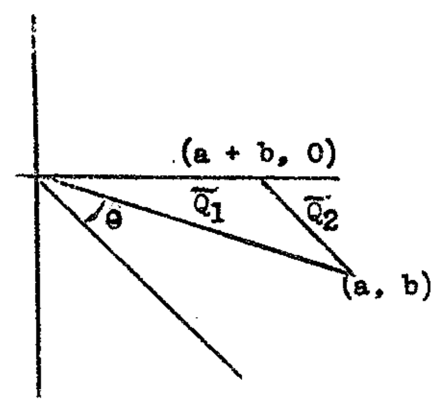
Now we may consider what the axioms imply with respect to $C(\tilde{Q})$. We have already discussed an implication of consequence 1. Consequences 3 and 4 imply that $C(\tilde{Q})$ is invariant with respect to a translation and an expansion or contraction about the origin which makes the transformed set a $\tilde{Q}$. The transformed set is obviously a closed convex set. However one must be sure that x_1, x_2 still lie in the interval of attainable payoffs.

The problem where $\tilde{Q}$ has only one point has by axiom V that point for a solution. Consider the problem $\tilde{Q}$ of the line connecting (0, 0) to (a, b). [We denote a line connecting two points x and y by [x to y]]. If the line is in the closed first quadrant (a, b) P (at, bt), $0 \leq t < 1$ and thus by axioms 1 and 3, $\{(a, b)\} = C(\tilde{Q})$. Similarly if the line is the closed third quadrant $\{(0, 0)\} = C(\tilde{Q})$.

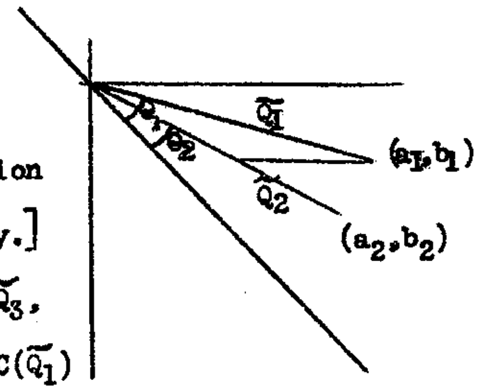
We associate with the line x to y an angle θ which the vector from x to y makes with the vector from (0, 0) to (1, −1). Thus we have yet to consider the lines where $-45 < \theta < 45$, and $135 < \theta < 225$. The cases where $135 < \theta < 225$ can be reduced to those where $-45 < \theta < 45$ by translating so that (a, b) goes into (0, 0) after contracting if necessary to keep within the bounds of attainability. The cases where $-45^\circ < \theta \leq 0$ can be reduced to the case of $0 \leq \theta < 45^\circ$ by translating so that (a, b) goes into (0, 0) and relfecting with respect to the line $x_1 = x_2$ after carrying through the necessary contractions.

Suppose that an interior point of the line is in $C(\tilde{Q})$. By consequence 2, the whole line is in $C(\tilde{Q})$. Suppose both end points of the line are in $C(\tilde{Q})$. Then by contraction (ta, tb) is in $C(\tilde{Q}_1)$ where $\tilde{Q}_1$ is the line (0, 0) to (ta, tb), $0 < t < 1$. By translating (ta, tb) to the origin (ta, tb) is in $C(\tilde{Q}_2)$ where $\tilde{Q}_2$ is the line (ta, tb) to (a, b). Axiom 7 implies that (ta, tb) is in $C(\tilde{Q})$ since $\tilde{Q} = \tilde{Q}_1 + \tilde{Q}_2$. Hence, $C(\tilde{Q})$ consists of one end point or of the whole line. Indeed the above type of argument can be used to show that if $\theta = 0$, $C(\tilde{Q})$ consists of the whole line.

Suppose that $0 < \theta < 45^\circ$. Consider the problem $\tilde{Q}_1$ consisting of the triangle whose vertices are (0, 0), (a, b), (a + b, 0). Let $\tilde{Q}_2$ be the problem of the line (a + b, 0) to (a, b) and $\tilde{Q}_3$ of the triangle except for the line $\tilde{Q}_2$. $\tilde{Q}_1 = \tilde{Q}_2 + \tilde{Q}_3$ and for every d_3 of $\tilde{Q}_3$ there is a d_2 of Q_2 such that $d_2 P d_3$. Thus axiom 6 implies that $C(\tilde{Q}_1) = C(\tilde{Q}_2)$. By translation within the bounds of attainability $C(\tilde{Q}_2)$ consists of the whole line $\tilde{Q}_2$ and $(a, b) \in C(\tilde{Q}_1)$. Hence $(a, b) \in C(\tilde{Q})$ by axiom 5.



Suppose that $\tilde{Q}_1$ and $\tilde{Q}_2$ are two lines from (0, 0) to (a_1, b_1), (a_2, b_2) respectively and $0 < \theta_2 < \theta_1 < 45^\circ$ and furthermore that the entire line $\tilde{Q}_1$ lies



in $C(\widetilde{Q}_1)$. Contract or expand $\widetilde{Q}_2$ so that b_2 goes into b_1. Let this new line be $\widetilde{Q}_3$. [This operation keeps the line within the bounds of attainability.] By axiom 6, if $\widetilde{Q}$ is the triangle bounded by $\widetilde{Q}_1$, $\widetilde{Q}_3$, and the horizontal line connecting them, $C(\widetilde{Q}) = C(\widetilde{Q}_1)$ which contains all strategies of $\widetilde{Q}_1$. Thus (0, 0) is in $C(\widetilde{Q})$ and by axiom 5 in $C(\widetilde{Q}_2)$. Since both end points of $\widetilde{Q}_2$ are then in $C(\widetilde{Q}_2)$, the whole line is 1. Thus we have

THEOREM 1. Corresponding to a solution $C(Q)$ defined on G the set of two column problems which are mixed problems of problems with a finite number of pure strategies with attainable payoffs, there is an angle θ_0, $0 \leq \theta_0 \leq 45$ and a case (inclusive or exclusive) so that if $\widetilde{Q}$ the line from x to y has an angle θ, then $C(Q)$ is

1)	The whole line if	$-\theta_0 < \theta < \theta_0$	exclusive case
		$-\theta_0 \leq \theta \leq \theta_0$	inclusive case
2)	y if	$\theta_0 \leq \theta \leq 180^\circ - \theta_0$	exclusive case
		$\theta_0 < \theta < 180^\circ - \theta_0$	inclusive case
3)	The whole line if	$180^\circ - \theta_0 < \theta < 180^\circ + \theta_0$	exclusive case
		$180^\circ - \theta_0 \leq \theta \leq 180^\circ + \theta_0$	inclusive case
4)	x if	$180^\circ + \theta_0 \leq \theta \leq 360^\circ - \theta_0$	exclusive case
		$180^\circ + \theta_0 < \theta < 360^\circ - \theta_0$	inclusive case

Also, if $\theta_0 = 0$, the case is the inclusive case and if $\theta_0 = 45^\circ$ it is the exclusive case.

Now consider any closed convex polygon $\widetilde{Q}$ with a finite number of vertices which lies in the region of attainable payoffs.

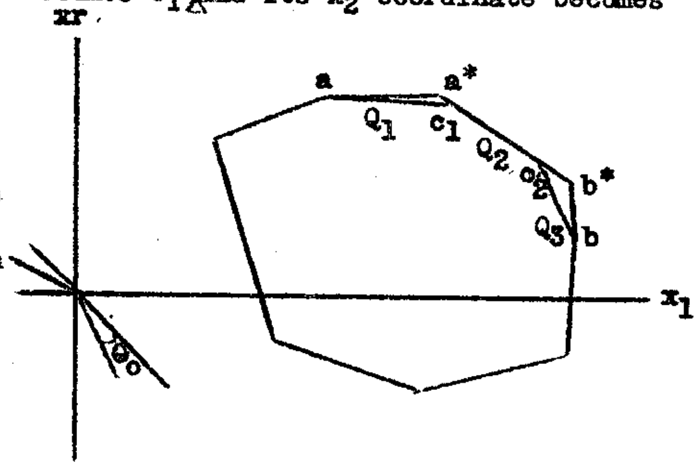
$$\text{Let } b_1 = \sup\left\{x_1 \mid (x_1, x_2) \in \tilde{Q}\right\}$$
$$b_2 = \sup\left\{x_2 \mid (b_1, x_2) \in \tilde{Q}\right\}$$
$$a_2 = \sup\left\{x_2 \mid (x_1, x_2) \in \tilde{Q}\right\}$$
$$a_1 = \sup\left\{x_1 \mid (x_1, a_2) \in \tilde{Q}\right\}$$

Traversing the boundary of $\tilde{Q}$ in a clockwise direction from a to b one passes the vertices $c_0 = a$, c_1, c_2, ..., c_{n-1}, $c_n = b$, let $\tilde{Q}_i$ be the line from c_{i-1} to c_i and θ_i. The corresponding angle of the line translated to the origin $45 > \theta_1 > \theta_2 > \ldots > \theta_n > -45^\circ$. Furthermore the angle of the line connecting any two points between c_{i-1} and c_j is between θ_i and θ_j (if the orientation is taken in the correct direction). Suppose $\tilde{Q}_i$ has an angle θ_i so that

$-\theta_o < \theta_i < -\theta_o$ in the exclusive case or

$-\theta_o \leq \theta_i \leq \theta_o$ in the inclusive case.

Then extend $\tilde{Q}_i$ until its x_1 coordinate becomes b_1 at point b^* and its x_2 coordinate becomes a_1 at a^*. Replace the boundary $\tilde{Q}_1$, $\tilde{Q}_2$ by the line $\tilde{Q}_1^* = (a, a^*)$ (it may be only a point). The line $\tilde{Q}^* = (a^*, b^*)$ and the line (b^*, b). Thus $\tilde{Q}$ has been increased to a new problem $\tilde{Q}_o^*$; $\tilde{Q}_o^*$ is the representation of a problem in G. $\tilde{Q}_o^* = \tilde{Q}^* + \tilde{Q}^{**}$ where $\tilde{Q}^{**}$ is the new problem except for $\tilde{Q}^*$.



To every point d^{**} of $\tilde{Q}^{**}$ there is a d^* of $\tilde{Q}^*$ so that d^*Pd^{**}. Thus $C(Q_o^*) = C(\tilde{Q}^*)$ by axiom 6. By Theorem 1, $C(Q^*)$ contains $\tilde{Q}^*$, which contains Q_i. Thus by axiom 5 $\tilde{Q}_i$ is in $C(\tilde{Q})$.

If there is no $\tilde{Q}_i$ satisfying the above requirements, a similar proof can be used to show that $C(\tilde{Q})$ contains the point (x_1, x_2) for which $x_1 + x_2$ attains its maximum.

Conversely it can be shown that in this case $C(\widetilde{Q}) = \{(x_1, x_2)\}$ and in the case where there are $\widetilde{Q}_i$'s satisfying the above restrictions only the points on the $\widetilde{Q}_i$ are in the solution. Thus we have a complete characterization of the solution. To state it in an alternative fashion, we first consider the following. For every θ in $(-\theta_o, \theta_o)$ (this interval is closed in the inclusive case and open in the exclusive case) there is a p so that $0 \leq p \leq 1$ and $\frac{p}{1 - p} = \tan(\theta + 45)$. Corresponding to the interval $(-\theta_o, \theta_o)$ there is an interval $(p_o, 1 - p_o)$.

THEOREM 2. If $\widetilde{G}$ is the set of representations of G, and G is the class of all problems with attainable payoffs which are the mixed problems of problems with a finite number of rows, then if G is rationally solvable by C(Q) a point a is in $C(\widetilde{Q})$ if and only if a is in $\widetilde{Q}$ and for some p in $(p_o, 1 - p_o)$, $pa_1 + (1 - p)a_2 \geq px_1 + (1 - p)x_2$ for all x in $\widetilde{Q}$.

This theorem is equivalent to stating that every problem of Q of G may be replaced by $Q^* = (u, D, S^*)$ where S^* is the set of mixed strategies of S where one state has probability p and the other 1 - p, p in $(p_o, 1 - p_o)$. Then C(Q) is the set of admissible strategies of Q^*.

THEOREM 3. If G of Theorem 2 is rationally solvable by C(Q), $\theta_o = 0$, (and the case is inclusive). This follows immediately from axiom 12.

The solution of the two column problems can be interpreted in a slightly different fashion. C(Q) is the set of strategies which minimize the maximum regret in comparisons at a time. If $a_1 + a_2 \geq x_1 + x_2$, then $a_1 - x_1 \geq x_2 - a_2$ or $a_2 - x_2 \geq a_1 - x_1$, and one of these statements is equivalent to the fact that in comparing (a_1, a_2) with (x_1, x_2), a_1, a_2 minimizes the maximum regret.

Theorem 2 can be extended to the case where G is the class of all problems where the representations are closed bounded convex sets within the region of attainable payoffs. This in turn is extendable to the case where the representations are convex in the region of attainable payoffs and a nun-null subset of the solution of the closure of the convex set is in the original set.

Theorems 1, 2, 3 and consequences 1, 2, 3, 4 made use of all the axioms except 8 which evidently doesn't apply unless we included the one column case in G. It is easy to prove that the solution for any $(p_0, 1 - p_0)$ would satisfy all axioms except 12, and the solution for $p_0 = \frac{1}{2}$ (inclusive case) satisfies axiom 12. Thus we have

THEOREM 4: G of Theorems 2 and 3 is rationally solvable by C(Q) where C(Q) is the set of strategies for which the sum of the payoffs attain the maximum. There is no other "rational" solution of G.

The arguments used here seem to be easily extendable to the "general" n column case. However if axiom 12 were abolished, they would still indicate a method which would apply to the class of problems with no more than n columns.

At this point a criticism of at least the use of certain G's should be made. A constructive criticism would be especially appreciated.