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Identification Problems with Serially Correlated Disturbances I

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Suppose we have a stochastic difference equation

(1) $$\sum_{\tau=0}^{n-1} B_\tau y'_{t-\tau} = u'_t$$

with u'_t satisfying

(2) $$u'_t - Pu'_{t-1} = v'_t$$

with independent v . Substituting (2) in (1), we obtain

(3) $$\sum_{\tau=0}^{n} B^* y'_{t-\tau} = v'_t,$$

with

(4) $$B^*_\tau = B_\tau - PB_{\tau-1}.$$

We know that the matrices $B_0^{-1}B^*_\tau$ are identified. Let us assume $B_0 = I$. We would like like to obtain identification conditions on B_τ.

Multiply (4) by $P^{n-\tau}$ and sum. We obtain

(5) $$\sum_\tau P^{n-\tau} B^*_\tau = \sum\left(P^{n-\tau} B_\tau - P^{n-\tau+1} B_{\tau-1}\right) = 0.$$

Conversely, if P satisfies (5), then

(6) $$B_\tau = B^*_\tau + PB_{\tau-1}, \qquad \tau \geq 1,$$

satisfy (4). Thus we must study the solutions of (5).

According to MacDuffee*[1], consider the polynomial

(7) $$f(\lambda) = \left|\sum \lambda^{n-\tau} B^*_\tau\right|.$$

The polynomial $f(\lambda)$ is of degree nG. It is shown in [1] that $f(P) = 0$. We shall here give a demonstration of that theorem which yields all solutions of (7).

*This reference was pointed out by Professor MacLane.

Let $P = V^{-1}\Lambda V$, V non-singular, Λ in classical canonical form.

Let $A_1, \ldots, A_\gamma$ be the decomposition of Λ corresponding to distinct characteristic roots of P. Then each A_i is essentially a polynomial in Λ. Also we have

(8) $$\sum A_i^{n-\tau} V_i B_\tau^* = 0.$$

Now $A_i = \lambda_i I + C_i$, C_i nilpotent.

Let $g(\lambda)$ be the matrix polynomial

(9) $$g_i(\lambda) = \sum \lambda^{n-\tau} V_i B_\tau^*.$$

Then

(10) $$\sum \frac{1}{j!} C_i^j g_i^{(j)}(\lambda_i) = 0.$$

Let $h(\lambda)$ be the matrix polynomial

(11) $$h(\lambda) = \sum \lambda^{n-\tau} V B_\tau^* V^{-1}.$$

Then we see that λ_i is a root of $f(\lambda) = |h(\lambda)|$ to at least its degree in c.f.P , and consequently $|\lambda I-P|$ divides $f(\lambda)$.

We know that A_i is a root of $g_i(\lambda)$. Consider the matrix polynomial

(12) $$k(\lambda) = \sum \lambda^{n-\tau} B_\tau^*.$$

Let λ_i be a root of $f(\lambda)$. Reduce $k(\lambda_i)$ to classical canonical form. Let W_i mediate this reduction, i.e., we take $W_i k(\lambda) W_i^{-1}$. This is

(13) $$W_i k(\lambda) W_i^{-1} = \sum \lambda^{n-\tau} W_i B_\tau^* W_i^{-1} = \begin{pmatrix} E_{1j} & 0 \\ 0 & E_{2j} \end{pmatrix},$$

where E_{1j} has 0's down and below the main diagonal. Then we see that V_i is a block of some W_i corresponding to some block of E_{1j} and conversely, if V_i is a block of some W_i, we can find exactly one suitable C_i.

Consequently, we may construct all solutions of (5) as follows:

Let μ be a root of $f(\lambda) = 0$; form $k(\mu)$; select a possible block V_i and C_i, hence A_i; put the V_i's together to form V, and the A_i's together to form Λ; then calculate $P = V^{-1}\Lambda V$.

Corollary: Equation (5) has a finite number of solutions if and only if the rank of $k(\lambda)$ is at least G-1 for all λ.

REFERENCE

1. C. C. MacDuffee, The Theory of Matrices, Ergebnisse der Mathematik und ihrer Granzgebiete, Berlin: Julius Springer, 1933, pp. ii, 5.