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The Effect of a Small Error in Observations on the Limited Information Estimates of a Stochastic Equation

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1. Summary of Calculations $\quad \beta y_t^{\prime *} + \gamma z_t^{\prime *} = u_t^{\prime}$

(1) M_{xx} = matrix with general term $T\sum_t x_{it}x_{jt} - (\sum x_{it})(\sum x_{jt})$

(2) $P^{\pm} = M_{y*z*}M_{z*z*}^{-1}$

(3) $W_{y*y*} = M_{y*y*} - M_{y*z}M_{zz}^{-1}M_{zy*}$

(4) $B_{y*y*} = M_{y*z}M_{zz}^{-1}M_{zy*} - M_{y*z*}M_{z*z*}^{-1}M_{z*y*}$

(5) λ = largest characteristic value of $\left|W_{y*y*} - \lambda B_{y*y*}\right|$

(6) β = corresponding characteristic vector

(7) $\gamma = -\beta P^{\pm}$

2. The Effect of an Error on M_{xx}

Suppose x_{it} is altered by an amount ρ_{it} and hence M_{xx} by μ_{xx}

then

$$M_{xx} + \mu_{xx} = T\sum_t(x_{it} + \rho_{it})(x_{jt} + \rho_{jt}) - \sum_t(x_{it} + \rho_{it})\sum_t(x_{jt} + \rho_{jt})$$

$$\mu_{xx} = T\left\{\sum_t x_{it}\rho_{jt} + \sum_t x_{jt}\rho_{it} + \sum_t \rho_{it}\rho_{jt}\right\} -$$

$$- \left\{\sum_t \rho_{it}\sum_t x_{jt} + \sum_t \rho_{jt}\sum_t x_{it} + \sum_t \rho_{it}\sum_t \rho_{jt}\right\}$$

If the error is "small," we can neglect higher order terms and

we get

$$\mu_{xx} = T\left\{\sum_t x_{it}\rho_{jt} + \sum_t x_{jt}\rho_{it}\right\} - \left\{\left(\sum_t \rho_{it}\sum_t x_{jt}\right) + \left(\sum_t \rho_{jt}\sum_t x_{it}\right)\right\} \tag{8a}$$

In any case if there is only one error, say in $x_{i\tau}$, then

$$\mu_{xx} = \left\{T(x_{j\tau}\rho_{i\tau}) - \rho_{i\tau}\sum_t x_{jt}\right\}\left[1 + \delta_{ij}\right]$$

(8b) $$\mu_{xx} = T\rho_{i\tau}\left[1 + \delta_{ij}\right]\left[x_{j\tau} - \bar{x}_j\right] \quad ; \bar{x}_j = \frac{1}{T}\sum_t x_{jt}$$

(8c) $$\mu_{xx} = \mu_1 + \mu_2$$ where μ_1 is a matrix consisting only of zeros except for one row and $\mu_2 = \mu_1'$

3. The Effect of a Small Error in M_{xx} on M^{-1}_{z*z*}, $P^{\pm}$, and $M_{y*z*} M^{-1}_{z*z*} M_{z*y*}$

$$(M_{z*z*} + \mu_{z*z*})^{-1} = \left\{(I + \mu_{z*z*} M^{-1}_{z*z*})M_{z*z*}\right\}^{-1}$$
$$= M^{-1}_{z*z*}\left\{I + \mu_{z*z*}M^{-1}_{z*z*}\right\}^{-1}$$
$$\approx M^{-1}_{z*z*}\left[I - \mu_{z*z*}M^{-1}_{z*z*}\right]$$

(9) $$(M_{z*z*} + \mu_{z*z*})^{-1} \approx M^{-1}_{z*z*} - M^{-1}_{z*z*}\mu_{z*z*}M^{-1}_{z*z*}$$

$$(M_{y*z*} + \mu_{y*z*})(M_{z*z*} + \mu_{z*z*})^{-1} \approx M_{y*z*}M^{-1}_{z*z*} + \mu_{y*z*}M^{-1}_{z*z*} - M_{y*z*}M^{-1}_{z*z*}\mu_{z*z*}M^{-1}_{z*z*}$$

Thus if $\Pi^{\pm}$ represents the change in $P^{\pm}$ we have

(10) $$\Pi^{\pm} \approx \mu_{y*z*}M_{z*z*} - P^{\pm}\mu_{z*z*}M^{-1}_{z*z*}$$

Now

$$(M_{y*z*} + \mu_{y*z*})(M_{z*z*} + \mu_{z*z*})^{-1}(M_{z*y*} + \mu_{z*y*})$$
$$\approx M_{y*z*}M^{-1}_{z*z*}M_{z*y*} + \mu_{y*z*}M^{-1}_{z*z*}M_{z*y*} + M_{y*z*}M^{-1}_{z*z*}\mu_{z*y*}$$
$$- M_{y*z*}M^{-1}_{z*z*}\mu_{z*z*}M^{-1}_{z*z*}M_{z*y*}$$

Thus if Θ_{y*y*} represents the change in $M_{y*z*}M^{-1}_{z*z*}M_{z*y*}$

(11) $$\Theta_{y*y*} = \mu_{y*z*}P^{\pm\prime} + P^{\pm}\mu_{z*y*} - P^{\pm}\mu_{z*z*}P^{\pm\prime}$$

Similarly if Ω_{y*y*} is the change in $W_{y*y*} = M_{y*y*} - M_{y*z}M^{-1}_{zz}M_{zy*}$

(12) $$\Omega_{y*y*} = \mu_{y*z}M^{-1}_{zz}M_{zy*} + M_{y*z}M^{-1}_{zz}\mu_{zy*} - M_{y*z}M^{-1}_{zz}\mu_{zz}M^{-1}_{zz}M_{zy*}$$

The ideas of (11) and (12) permit us to obtain Δ_{y*y*} the change in B_{y*y*}.

4. The Effect of Small Changes in $W_{y^*y^*}$ and $B_{y^*y^*}$ on λ and β.

We shall call the changes in λ and β, ε and ζ respectively. We also assume that the first coordinate of β will be kept fixed, i. e. $\zeta = (\zeta_1, \zeta_{-1})$ where $\zeta_1 = 0$. (Normalization on this first coordinate is permissible as long as it isn't zero in the original or final state; this possibility is quite slight and even if it occurred we could normalize on some other coordinate of β.)

We have

$$\left\{(W_{y^*y^*} + \Omega_{y^*y^*}) - (\lambda + \varepsilon)(B_{y^*y^*} + \Delta_{y^*y^*})\right\}(\beta + \zeta)' = 0 \tag{13}$$

whence neglecting second order terms and using the fact that $(W_{y^*y^*} - \lambda B_{y^*y^*})\beta' = 0$ we have

$$\Omega_{y^*y^*}\beta' - \varepsilon B_{y^*y^*}\beta' - \lambda\Delta_{y^*y^*}\beta' + W_{y^*y^*}\zeta' - \lambda B_{y^*y^*}\zeta' = 0$$

$$(W_{y^*y^*} - \lambda B_{y^*y^*})\zeta' + (\Omega_{y^*y^*} - \varepsilon B_{y^*y^*} - \lambda\Delta_{y^*y^*})\beta' = 0 \tag{14}$$

Now pre-multiply equation (14) by β. By the symmetry of $W_{y^*y^*} - \lambda B_{y^*y^*}$ we have $\beta(W_{y^*y^*} - \lambda B_{y^*y^*}) = 0$ and hence

$$\varepsilon = \frac{\beta(\Omega_{y^*y^*} - \lambda\Delta_{y^*y^*})\beta'}{\beta B_{y^*y^*}\beta'} \tag{15}$$

and

$$(W_{y^*y^*} - \lambda B_{y^*y^*})\zeta' = \varepsilon B_{y^*y^*}\beta' - (\Omega_{y^*y^*} - \lambda\Delta_{y^*y^*})\beta'$$

Now we can partition the matrices $W_{y^*y^*}$ and $B_{y^*y^*}$ etc. according to the following scheme

$$B_{y^*y^*} = \begin{vmatrix} B_{y_1^*y_1^*} & B_{y_1^*y_{-1}^*} \\ B_{y_{-1}^*y_1^*} & B_{y_{-1}^*y_{-1}^*} \end{vmatrix}$$

We assume that $W_{y^*_{-1}y^*_{-1}} - \lambda B_{y^*_{-1}y^*_{-1}}$ is nonsingular. In that event, we can solve for ζ_{-1}

(16) $$\begin{cases} \zeta_1 = 0 \\ \zeta_{-1} = (W_{y^*_{-1}y^*_{-1}} - \lambda B_{y^*_{-1}y^*_{-1}})^{-1}\left\{\varepsilon B_{y^*_{-1}y^*}\beta' - (\Omega_{y^*_{-1}y^*} - \lambda \Delta_{y^*_{-1}y^*})\beta'\right\} \end{cases}$$

5. **Final Results**

ζ represents the change of β; we have only to calculate η, the change of γ.

$$(\gamma + \eta_1) = -(\beta + \zeta)(P^{\pm} + \Pi^{\pm})$$

(17) $$\eta_1 = -(\zeta P^{\pm} + \beta \Pi^{\pm})$$

It must be kept in mind that these are the effects of a "small" error, i.e. we have actually calculated the first order terms of the effects of the error. Section Three indicates how to get the effect of the error on the least squares estimates and the explained variance W_{yy}.