

Note on Computation of Sampling Variances and  
Covariances in the Case of Reduced Form Estimation

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The following will indicate how to compute the sampling variances and covariances after "limited information"\* estimation has been completed. It is based on Anderson and Rubin's, "Estimation of the Parameters of a Single Stochastic Difference Equation in a Complete System." We shall use the same notation, and all references are to this mimeographed article.

According to footnote 17, we may in 7.96-98 omit any column or row, instead of only the N-th. It is convenient to omit the 1st when we normalize

$$(1) \quad \underline{\beta} = (1, \beta_2 \dots \beta_{+1})$$

as is usually the case. 7.96 then becomes

$$(2) \quad \sigma(\hat{\beta}, \hat{\beta}) = c(\underline{1} - \beta' \psi)_{.;1} \left[ \begin{matrix} \theta \\ \end{matrix} \right]_{1;1}^{-1} (\underline{1} - \psi' \beta)_{1;.(}$$

where the subscript  $);1($  to any matrix A indicates that no row and the first column of A is to be omitted;  $);1;1($  means that both first column and row are omitted. In (2) we have (from 3.46)

$$(3) \quad c = \hat{\beta} \Omega \hat{\beta}' = (1 + v) \hat{\beta} W \hat{\beta}'$$

where v is obtained from 4.17 and W from 4.7, while  $\hat{\beta}$  is the result of the estimation after normalization.

In the usual case (1) the matrix  $\zeta$  which describes the normalization rule is

$$\zeta = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

\* This method was formerly called the "reduced form" method.

so that it follows from 7.48 that

$$\psi = (1 \dots \beta_{+1}) \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} = (1, 0, \dots, 0).$$

Therefore we have in (2)

$$(1 - \beta' \psi)_{1;1} = \left[ 1 - \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \right]_{1;1} = \begin{pmatrix} 0 & \dots & \dots & 0 \\ 1 & \dots & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 1 \end{pmatrix}$$

This reduces (2) to

$$(4) \sigma(\hat{\beta}' \hat{\beta}) = \sigma \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left[ \Theta \right]_{1;1}^{-1} (0) = \sigma \begin{pmatrix} 0 & 0 \\ 0 & (\Theta)_{1;1}^{-1} \end{pmatrix}$$

where  $\Theta$  is estimated as follows.

From 7.40\* (corrected):

$$(5) \hat{\Theta} = P^{**} M_{z^0 z^0} P^{**'} - \frac{1}{c} \frac{v}{1+v} \Omega \hat{\beta}' \hat{\beta} \Omega$$

Using (3) and 3.41 (in which  $1 = 1 + v$ ) we write for the last term

$$\frac{1}{(1+v) \hat{\beta}' W \hat{\beta}} \cdot \frac{v}{1+v} (1+v)^2 W \hat{\beta}' \hat{\beta} W \text{ so that we compute } \hat{\Theta} \text{ from}$$

$$(6) \hat{\Theta} = P^{**} M_{z^0 z^0} P^{**'} - v \frac{W \hat{\beta}' \hat{\beta} W}{\hat{\beta}' W \hat{\beta}}$$

which can be done directly from the intermediate steps in the "limited information" estimation. Having  $\sigma(\hat{\beta}' \hat{\beta})$  we obtain the remaining variances and covariances from

$$(7) \sigma(\hat{\beta}' \hat{\gamma}^*) = \sigma(\hat{\beta}' \hat{\beta}) P^*$$

\* In T.W. Anderson and H. Rubin, "Outline of Computational Procedure for the Reduced Form Method," the matrix  $M_{z^0 z^0}$  is called N. In (7) we have  $P^*$ , for which Anderson and Rubin use the notation  $\tilde{P}^*$ .

$$(8) \sigma(\hat{\gamma}^*{}' \hat{\gamma}^*) = P' \sigma(\hat{\beta}' \hat{\gamma}^*) + c(M_{z^*z^*})^{-1}$$

Once the "limited information" estimation is completed, the outline for the computation of sampling variances and covariances is therefore as follows:

- (a) compute  $c$  from (3)
- (b) compute  $\Theta$  from (6); omit the first row and column [ to get  $\hat{\Theta}_{1;1}$  and invert  $\hat{\Theta}_{1;1}$  ]
- (c) estimate  $\sigma(\hat{\beta}' \hat{\beta})$  from (4)
- (d) estimate  $\sigma(\hat{\beta}' \hat{\gamma}^*)$  and  $\sigma(\hat{\gamma}^*{}' \hat{\gamma}^*)$  from (7) and (8)