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Transformations of Dual Linear Programs

Summary

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Dual Linear Programs (with "mixed" constraints):

To maximize  $c_1x_1 + \dots + c_nx_n - d$  constrained by

$$a_{11}x_1 + \dots + a_{in}x_n - b_i \begin{cases} \leq 0 & \text{for each } i \text{ in } M_1 \\ = 0 & \text{for each } i \text{ in } M_2 \end{cases}$$

$$x_j \begin{cases} \text{nonnegative for each } j \text{ in } N_1 \\ \text{unrestricted for each } j \text{ in } N_2. \end{cases}$$

To minimize  $u_1b_1 + \dots + u_mb_m - d$  constrained by

$$u_1a_{1j} + \dots + u_ma_{mj} - c_j \begin{cases} \geq 0 & \text{for each } j \text{ in } N_1 \\ = 0 & \text{for each } j \text{ in } N_2 \end{cases}$$

$$u_i \begin{cases} \text{nonnegative for each } i \text{ in } M_1 \\ \text{unrestricted for each } i \text{ in } M_2. \end{cases}$$

Note.  $M_1$  and  $M_2$  are complementary subsets of the set  $M = [1, 2, \dots, m]$ ;

$N_1$  and  $N_2$  are complementary subsets of the set  $N = [1, 2, \dots, n]$ .

Transformations (reducing the number of constraint equations):

Type 1. Assume  $a_{rs} \neq 0$ ,  $r$  in  $M_2$ ,  $s$  in  $N_2$ . Then the unrestricted variables  $x_s$  and  $u_r$  can be eliminated through

$$(1) \quad x_s = \frac{1}{a_{rs}} (b_r - \sum_{j \neq s} a_{rj} x_j)$$

$$(2) \quad u_r = \frac{1}{a_{rs}} (c_s - \sum_{i \neq r} u_i a_{is}).$$

In this way one passes to equivalent dual programs in which the coefficients are

$$(3) \quad \left\{ \begin{array}{l} \bar{a}_{ij} = a_{ij} - \frac{a_{rj} a_{is}}{a_{rs}}, \quad \bar{b}_i = b_i - \frac{b_r a_{is}}{a_{rs}} \\ \bar{c}_j = c_j - \frac{a_{rj} c_s}{a_{rs}}, \quad \bar{d} = d - \frac{b_r c_s}{a_{rs}} \end{array} \right\} \quad (i \neq r, j \neq s)$$

with  $\bar{M}_1 = M_1$ ,  $\bar{M}_2 = M_2 - [r]$ ,  $\bar{N}_1 = N_1$ ,  $\bar{N}_2 = N_2 - [s]$ .

Type 2. Assume  $a_{rs} \neq 0$ ,  $r$  in  $M_2$ ,  $s$  in  $N_1$ . Then the non-negative variable  $x_s$  can be eliminated through (1) and the unrestricted variable  $u_r$  replaced by the non-negative variable

$$\bar{u}_r = \sum u_i a_{is} - c_s.$$

In this way one passes to equivalent dual programs in which the coefficients are given by (3) and by

$$\bar{a}_{rj} = \frac{a_{rj}}{a_{rs}}, \quad \bar{b}_r = \frac{b_r}{a_{rs}} \quad (j \neq s)$$

with  $\bar{M}_1 = M_1 + [r]$ ,  $\bar{M}_2 = M_2 - [r]$ ,  $\bar{N}_1 = N_1 - [s]$ ,  $\bar{N}_2 = N_2$ .

Type 3. Assume  $a_{rs} \neq 0$ ,  $r$  in  $M_1$ ,  $s$  in  $N_2$ . Then the non-negative variable  $u_r$  can be eliminated through (2) and the unrestricted variable  $x_s$  replaced by the nonnegative variable

$$\bar{x}_r = b_r - \sum a_{rj} x_j .$$

In this way one passes to equivalent dual programs in which the coefficients are given by (3) and by

$$\bar{a}_{is} = - \frac{a_{is}}{a_{rs}}, \quad \bar{c}_s = - \frac{c_s}{a_{rs}} \quad (i \neq r)$$

with  $\bar{M}_1 = M_1 - [r]$ ,  $\bar{M}_2 = M_2$ ,  $\bar{N}_1 = N_1 + [s]$ ,  $\bar{N}_2 = N_2 - [s]$ .