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# Transformations of Dual Linear Programs

## Summary

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Dual Linear Programs (with "mixed" constraints):

To maximize $c_1x_1 + \dots + c_nx_n - d$ constrained by

$$a_{i1}x_1 + \dots + a_{in}x_n - b_i \begin{cases} \leqq 0 & \text{for each } i \text{ in } M_1 \\ = 0 & \text{for each } i \text{ in } M_2 \end{cases}$$

$$x_j \begin{cases} \text{nonnegative} & \text{for each } j \text{ in } N_1 \\ \text{unrestricted} & \text{for each } j \text{ in } N_2. \end{cases}$$

To minimize $u_1b_1 + \dots + u_mb_m - d$ constrained by

$$u_1a_{1j} + \dots + u_ma_{mj} - c_j \begin{cases} \geqq 0 & \text{for each } j \text{ in } N_1 \\ = 0 & \text{for each } j \text{ in } N_2 \end{cases}$$

$$u_i \begin{cases} \text{nonnegative} & \text{for each } i \text{ in } M_1 \\ \text{unrestricted} & \text{for each } i \text{ in } M_2. \end{cases}$$

Note. $M_1$ and $M_2$ are complementary subsets of the set $M = [1, 2, \dots, m]$; $N_1$ and $N_2$ are complementary subsets of the set $N = [1, 2, \dots, n]$.

Transformations (reducing the number of constraint equations):

Type 1. Assume $a_{rs} \neq 0$, $r$ in $M_2$, $s$ in $N_2$. Then the unrestricted variables $x_s$ and $u_r$ can be eliminated through

(1) $$x_s = \frac{1}{a_{rs}}(b_r - \sum_{j \neq s} a_{rj}x_j)$$

(2) $$u_r = \frac{1}{a_{rs}}(c_s - \sum_{i \neq r} u_i a_{is}).$$

In this way one passes to equivalent dual programs in which the coefficients are

(3) $$\left\{\begin{array}{ll} \bar{a}_{ij} = a_{ij} - \dfrac{a_{rj}a_{is}}{a_{rs}}, & \bar{b}_i = b_i - \dfrac{b_r a_{is}}{a_{rs}} \\ \bar{c}_j = c_j - \dfrac{a_{rj}c_s}{a_{rs}}, & \bar{d} = d - \dfrac{b_r c_s}{a_{rs}} \end{array}\right\} \quad (i \neq r,\ j \neq s)$$

with $\bar{M}_1 = M_1$, $\bar{M}_2 = M_2 - [r]$, $\bar{N}_1 = N_1$, $\bar{N}_2 = N_2 - [s]$.

Type 2. Assume $a_{rs} \neq 0$, $r$ in $M_2$, $s$ in $N_1$. Then the non-negative variable $x_s$ can be eliminated through (1) and the unrestricted variable $u_r$ replaced by the non-negative variable

$$\bar{u}_r = \sum u_i a_{is} - c_s.$$

In this way one passes to equivalent dual programs in which the coefficients are given by (3) and by

$$\bar{a}_{rj} = \frac{a_{rj}}{a_{rs}}, \quad \bar{b}_r = \frac{b_r}{a_{rs}} \qquad (j \neq s)$$

with $\bar{M}_1 = M_1 + [r]$, $\bar{M}_2 = M_2 - [r]$, $\bar{N}_1 = N_1 - [s]$, $\bar{N}_2 = N_2$.

Type 3. Assume $a_{rs} \neq 0$, $r$ in $M_1$, $s$ in $N_2$. Then the non-negative variable $u_r$ can be eliminated through (2) and the unrestricted variable $x_s$ replaced by the nonnegative variable

$$\bar{x}_s = b_r - \sum a_{rj} x_j \ .$$

In this way one passes to equivalent dual programs in which the coefficients are given by (3) and by

$$\bar{a}_{is} = - \frac{a_{is}}{a_{rs}}, \qquad \bar{c}_s = - \frac{c_s}{a_{rs}} \qquad\qquad (i \neq r)$$

with $\bar{M}_1 = M_1 - [r]$, $\bar{M}_2 = M_2$, $\bar{N}_1 = N_1 + [s]$, $\bar{N}_2 = N_2 - [s]$.