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The Assignment Problem

by

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0. Introduction. Let $a = (a_{ij})$ be an n by n matrix of real numbers. To find a permutation (p_i) such that $\Sigma\, a_{ip_i}$ be as great as possible is the assignment problem. For name and history see [1], prefixing [2] and [3].

To compute $n!$ sums of n terms and to choose a greatest sum is mathematically trivial; there remains a typical problem of numerical analysis: to find the solution, or an approximation of it, in a fast and convenient way, even for large n, or for a large set of matrices.

The following account is divided into the sub-headings Normalization, Approximate solution, Vertex approach and face approach methods, The equidistribution problem, Related problems. The material in the next to last heading is also of purely mathematical interest; it and part of the other sections are new. A summary of geometrical concepts used is appended.

It is convenient to look at a as a point or vector in real n^2-space. Defining the permutation point $p = (p_{ij})$ by $p_{ij} = 1$ for $j = p_i$ and $= 0$ for $j \neq p_i$ our problem is to maximize $\Sigma_{ij} a_{ij} p_{ij}$ or equivalently $\cos\,(a, p)$. The linear subspace L_1 spanned by the $n!$ points p is clearly the $(n-1)^2$-space defined by $\Sigma_i a_{ij} = \Sigma_j a_{ij} = 1$; its parallel L_0 through 0 is defined by $\Sigma_i a_{ij} = \Sigma_j a_{ij} = 0$. The orthogonal complement is the $(2n-1)$-space L'_0 consisting of all $a = (a_{ij})$, $a_{ij} = \lambda_i + \mu_j$, and meets L_1 at the point c, $c_{ij} = 1/n$, centroid of the points p. The permutation polyhedron or convex

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hull P of the points p is ([4]) given in L_1 by its n^2 faces $a_{ij} \geq 0$ (just for n = 1 and 2 the number of faces is 0 and 2, see section 4); its points are the doubly-stochastic matrices.

1. Normalization. Among transformations that do not change the solving permutation we mention:

1. Replacing a_{ij} by $e^{a_{ij}}$ and Σ by Π leads to the problem of finding the absolutely greatest term of a determinant. The problem of finding the, without qualification, greatest term (for $n \geq 3$ necessarily ≥ 0) is slightly different.
2. Replacing a by -a leads to the problem of finding a smallest permutation sum.
3. We may replace a by λa, $\lambda > 0$. Hence all elements can be supposed to be absolutely ≤ 1.
4. An arbitrary constant may be added to all elements of the same row or column.

By the last device (that is, adding an $\in L_0'$) we can achieve one of the following preconditionings.

41. Make all elements non-negative, or positive.

42. Make all elements non-negative, and have at least one zero in every row; thereafter also in every column. (Alternating between rows and columns will take, on the average, more steps.) If we now find a permutation formed by zeros (this, the ideal assignment problem [7], may itself not be easy) it will give a least permutation sum. In any event the rows and columns can now (again, not always easily) be rearranged so that a subset of zeros lies on a monotone and convex curve connecting two corners, that is, it consists of $a_{11}, \ldots, a_{1m_1}, a_{2,m_1+1}, \ldots, a_{2,m_2}, \ldots, a_{k,m_k}, a_{n-n_\ell,n+1-\ell}, \ldots, a_{n-n_{\ell-1}+\ell,\ n+1-\ell}, \ldots, a_{n-n_1,n}, \ldots,$

$a_{n,n}$, where $m_1 \geq m_2 - m_1 \geq \dots \geq m_k - m_{k-1}$, $n_1 \geq n_2 - n_1 \geq \dots \geq n_\ell - n_{\ell-1}$, $k + \ell = n - n_\ell$, $m_k = n - \ell$. (Fig. 1). (The number of such curves in a square or rectangle is related to the partition functions of additive number theory.)

43. Make all elements of the first row and column zero. This normalization is unique but unsymmetric.

44. Make all row sums and column sums zero. This normalization (amounting to the decomposition $a + \epsilon L_0' = \epsilon L_0$) is unique, symmetric and still very simple. Alternatively, choose $a + \epsilon L_0' + c = \epsilon L_1$.

45. After 44, add a constant to all elements so as to make them non-negative, but at least one of them zero. Thereafter multiply by a constant so as either to make the maximum element and thus also the spread 1, or preferably so as to make the common value of all row sums and column sums 1, thereby obtaining an equivalent ϵP with maximum spread.

46. Make the spread max a_{ij} - min a_{ij} a minimum. It is easily seen that the horizontal spread $\max_i(\max a_{ij} - \min a_{ij})$ and the vertical spread are then also minimal. Or minimize the horizontal spread sum $\Sigma_i(\max_{ij} - \min a_{ij})$ or the vertical spread sum. These are special linear minimization problems the first of which allows a simple minimum criterion involving circuits.

2. <u>Approximate solution</u> (cf. [1]). For shortening some of the direct methods below it may be desired to find a lower bound for the greatest permutation sum. Any permutation sum is such a bound, and may solve a modified problem like approximating, in some sense, the greatest sum, or surpassing a certain threshold value.

$\Sigma_i \min a_{ij}$ is an easily computable lower bound for all permutation sums. The quality of any lower bound can be appraised by comparing with the upper bound $\Sigma_i \max a_{ij}$.

The fact that every permutation sum is itself a lower bound leads to the proposal [9] to choose permutations at random. To get more permutations per second examined one may change them gradually by interchanging each time two indices. It is of course advisable to use an inbuilt random number generation procedure [10]. As for further variants one can allow exchanges of three or more indices and/or restrict changes to those for which the sum is raised or not too much dropped.

Among any n complementary permutations $p^{(k)}$ (that is, $\Sigma_k p_{ij}^{(k)} = 1$) at least one sum is $\geq \Sigma_{ij} a_{ij/n}$ (for doubly-stochastic matrices, 1).

The greatest and smallest permutation sum may, of course, have common elements.

2.1. A good lower bound, and one that is often naively surmised to be the solution, is $s = \Sigma a_{ip_i}$ where $a_{ij_i} = \max_{j \neq p_1, \cdots, p_{i-1}} a_{ij}$. For doubly-stochastic matrices $s = \Sigma a_{ij_i} \geq \Sigma_{ip_n} = 1$.

The bound s depends on the arrangement of rows and, for a certain arrangement (and choice of max a_{ij} in case of ambiguity), does yield the solution. For let Σa_{ip_i} be a greatest permutation sum. Have row i precede row j if $a_{jp_i} > a_{jp_j}$. This is feasible, since a contradiction like $a_{jp_i} > a_{jp_j}$, $a_{kp_j} > a_{kp_k}$, $a_{ip_k} > a_{ip_i}$ (a "circuit", Fig. 2) would show how to replace Σa_{ip_i} by a greater sum. Though the partial order indicated may uniquely determine the order of the rows, as in $\begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}$, or in $\frac{1}{72}\begin{pmatrix} 45 & 7 & 20 \\ 27 & 26 & 19 \\ 0 & 39 & 33 \end{pmatrix}$ or $\frac{1}{34}\begin{pmatrix} 12 & 9 & 9 & 4 \\ 13 & 12 & 0 & 9 \\ 0 & 13 & 12 & 9 \\ 9 & 0 & 13 & 12 \end{pmatrix}$ ϵP, it usually does not, which makes it seem worthwhile to try random arrangements of rows for s.

Since we see that every greatest permutation sum must contain a maximal element of some row, the sum can also be obtained by taking, after rearrangement of rows and columns, the greatest element of the first row, then, after deleting its column and row, the greatest element in the first column, alternating between rows and columns or using some other similar prescription.

A natural specialization of s, which is independent of the arrangement of rows and unaffected by transposing a, is $t = \sum a_{p_k q_k}$ where $a_{p_k q_k} = \max_{i \neq p_1, \cdots, p_{k-1}; j \neq q_1, \ldots, q_{k-1}} a_{ij}$. For positive a it is easy to see that $t > \sum_{ij} a_{ij}/(2n + 1)$. Though t, even for $a \in P$, need not give the solution, as shown by the two last numerical examples, t seems to be a good bet or start for $a \in L_1$. For $n \leqq 3$ and $a \in L_1$, every greatest permutation sum must contain $\max a_{ij}$; indeed, $1/3\ ((a_{11} + a_{22} + a_{33}) - (a_{12} + a_{23} + a_{31})) = a_{11} - a_{23} = a_{22} - a_{31} = a_{33} - a_{12}$ is easy to verify and entails just the statement made.

The t-procedure requires more comparisons than the s-procedure. The results of both procedures are, for a general two by two matrix, on the average equally good; for a three by three matrix reduced as in 1.42, the s-procedure gives, on the average, slightly better results.

[Insert Page 5a].

2.2. The matrix a may itself be replaced by an approximation or derived matrix a'. Any a' for which $a'_{ij} > a'_{ij'}$ if and only if $a_{ij} > a_{ij'}$ leads to the same s-permutations as a; retention of the >-relation over the whole matrix preserves the t-permutations. However these relations are of little relevance since addition of an $\epsilon L'_0$ destroys them. Not so for $a_{ij} + a_{i'j'} > a_{ij'} + a_{i'j}$, the assignment problem for every two by two submatrix; in fact, knowledge of this relation enables one, for $n = 3$, to determine either the greatest permutation sum, or at least the two greatest ones. [11] proposes to

An advantage of the t-procedure is that it also furnishes an upper bound. In fact, an easy induction shows that, for positive a, $2t$ surpasses every permutation sum [10']. This can be a better bound than $\sum_i \max a_{ij}$, e.g. for $a \epsilon P$ with $a_{i1} = 3/n$ or 0 as $i \leq$ or $> n/3$, $a_{i3} = 3/n$ or 0 as $i >$ or $\leq 2\,n/3$, $a_{i1} + a_{i2} + a_{i3} = 3/n$, $a_{ik} = 1/n$ for $k > 3$, $n/3$ an integer ≥ 3. The s-procedure has no corresponding property: choosing $a \epsilon P$ with all a_{ii}, a_{1i}, a_{in}, and $a_{1n} = 1/n$, all $a_{i,i-1} = 1 - 2/n$, all other $a_{ik} = 0$, we obtain $s = 1$ (this choice can be enforced by slightly altering a), but a greatest sum $n - 3 + 3/n$.

To obtain an upper bound it is not necessary to find all of t. In fact, the sum of the first k of the $2n$ terms $a_{p_1q_1}, a_{p_1q_1}, a_{p_2q_2}, a_{p_2q_2}, \ldots, a_{p_nq_n}, a_{p_nq_n}$ is $\geq$ every sum of k elements of a belonging to a permutation; this can again be verified by induction on n. For positive a the bound t' pertaining to $k = n$ is obviously $< 2t$.

However, t' can be $< u = \sum_i \max a_{ij}$ only for $n \geq 3$, and if $a \epsilon P$, only for $n \geq 5$, e.g. for

$$\frac{1}{225}\begin{pmatrix} \underline{63} & 6 & 52 & 52 & 52 \\ 0 & 57 & \underline{56} & 56 & 56 \\ 54 & \underline{54} & 39 & 39 & 39 \\ 54 & 54 & 39 & \underline{39} & 39 \\ 54 & 54 & 39 & 39 & \underline{39} \end{pmatrix}$$

with $u = \frac{282}{225}$, $t' = \frac{279}{225}$, $t = \frac{237}{225}$, greatest sum (underlined) $= \frac{251}{225}$. For $n = 4$ let $t' = 2a_{11} + 2a_{22}$, $a_{11} \geq a_{22}$. Clearly $t' \geq u$ if $\max_j a_{ij} > a_{i1}$ for more than one i; but if $\max_j a_{ij} = a_{i1}$ for $i \neq i_o$ then $u = \max_j a_{i_oj} + \sum_{j>1} a_{i_oj}$ is again $\leq t'$. Similarly for $n < 4$.

give up even more information by defining $a'_{ij} = \Sigma_{i',j'} 1$ for the above pairs i',j' (scoring 1/2 for equality); $2n^{-1}(n-1)^{-2}a'$ is ϵP. For $n \leq 3$ no information is lost, and $a'' = (a')' = a'$. An example with $a'' \neq a$ is

$$a = \frac{1}{41}\begin{pmatrix} 5 & 12 & 11 & 13 \\ 12 & 23 & 6 & 0 \\ 11 & 6 & 13 & 11 \\ 13 & 0 & 11 & 17 \end{pmatrix} \epsilon P, \quad a' = \begin{pmatrix} 1 & 6 & 5 & 6 \\ 6 & 9 & 3 & 0 \\ 5 & 3 & 6 & 4 \\ 6 & 0 & 4 & 8 \end{pmatrix}, \quad a'' = a''' = \begin{pmatrix} 0 & 7 & 5 & 6 \\ 7 & 8 & 3 & 0 \\ 5 & 3 & 6 & 4 \\ 6 & 0 & 4 & 8 \end{pmatrix}.$$

2.3. The very crudest approximation $n \max a_{ik}$ may still have a great probability of being correct. Consider a matrix a of non-negative elements, each of which, independently of the others, has a probability $1 - q > 0$ of being 0. What is the probability $1 - q'$ that the least sum is $n \min a_{ik} = 0$? It is known [7] that the least sum is 0 except if there exists a "block" of non-zero elements filling the intersections of some i rows and $n - i + 1$ columns, $1 \leq i \leq n$. The probability for a block at a given location is $q^{i(n-i+1)}$, and the probability of any i-block occurring is therefore at most $q_i = \binom{n}{i}\binom{n}{i-1} q^{i(n-i+1)}$. Hence $q' \leq \Sigma q_i$. Now we shall show that for $nq^{n/2} < 1/2$ (a fortiori for $n > -c_1(1 - q) \ln (1 - q) + c_2$ with universal constants c_1 and c_2), $q_i \leq q_1 = nq^n$, so that $q' \leq n^2q^n < 1/4$ and $q' \to 0$ for $n \to \infty$.

Since $q_i = q_{n+1-i}$ we can suppose $2i \leq n + 1$. We show that $q_{i+1}/q_i = (n - i + 1)(n - i)(i + 1)^{-1}i^{-1}q^{n-2i} < 1$ for $2i \leq n - 1$. For $n = 2i + 1$, this amounts to $l_i = (1 + 2i^{-1})^{i/2} < (4i + 2)^{i/(2i+1)} = r_i$, true for $i = 1$ and 2; for $i > 2$, $l_i \uparrow e < r_3 \leq r_i$. Now use induction on n for fixed i to prove $q_{i+1}/q_i < 1$, rewritten as

$$(1_n) \quad (1 - (i - 1)/n)^n(1 - i/n)^n(2n)^{4i} < 4^n i^n(i + 1)^n \quad .$$

It suffices to prove (l_{n+1}/l_n) where the three factors of the left side decrease with increasing n (convexity of the logarithmic function) whereas the right side remains $4i(i+1)$. Thus we have only to verify (l_{2i+2}/l_{2i+1}) which can be written $l_i' = (1 + 2/(i + 1))^{2i+2} < 16i(i + \frac{1}{2})^{-1}.(i + 1)^2$ $(i + \frac{1}{2})^{-1} (i + 2)^{-1}. (i + 1)^2 = r_i'$ and confirmed for $i = 1$; for $i > 1$, $l_i' \uparrow e^4 < r_2' \leq r_i'$.

This justification of the simplest estimate for the sum can be completed by a justification of the before-mentioned simplest procedures for finding a maximizing permutation, thus determining the circumstances under which the usual sweeping elimination of all more sophisticated methods is safe, a consideration of fundamental importance in practical application. Suppose $0 \leq a_{ij} < m$, a fixed integer, and define a' by $a_{ij}' = [a_{ij}]$, the greatest integer $\leq a_{ij}$. Assuming that a_{ij}', independently of every other element, is 0, 1, ..., m - 1 with equal probability 1/m, what are the expected value e of s (defined in 2.1, with min for max) and the probability q that s = 0? Evidently $q = \prod_i (1 - 1/m)^i)$, with a positive limit for $n \to \infty$. And since $e(n) - e(n - 1)$ is easily seen to be $(m - 1)m^{-n} + (m - 2)(2^n - 1) m^{-n} + \ldots + 2((m - 2)^n - (m - 3)^n)m^{-n} + 1((m - 1)^n - (m - 2)^n)m^{-n} = \sum_{k=1}^{m-1} (k/m)^n$, we find $e(n) = \sum_k (1 - k^n m^{-n})k/(m - k) \uparrow m \sum 1/k - (m - 1) \sim m \ln m$; e.g., $e(n) \uparrow 19 \frac{73}{252}$ for $m = 10$. As to e' and q', belonging to the corresponding t-procedure, we have $q' = \prod_i (1 - (1 - 1/m)^{i^2}) > q$ and, for $m = 2$, $e'(n) = 2^{-n^2} n + (1 - 2^{-n^2}) e'(n - 1)$ with $e' \uparrow .5785 \ldots$ versus $e = 1 - 2^{-n}$.

3. Vertex approach and face approach methods.

Checking all permutation sums of a non-negative matrix for a least sum can be shortened by dropping all those for which a partial sum surpasses an upper bound, whether otherwise obtained or formed by sums found during the procedure [K2].

A more methodic search for matrices consisting of integers [13] can be extended to general matrices as follows. Starting from some permutation p, an $\epsilon L_0'$ is added to a so that all elements in that permutation are 0. If all other elements are ≤ 0, p gives a greatest sum; if not choose a positive element α. Call its column c_1; the row in which c_1 and p meet, r_1; the columns of the non-negative elements in r_1, $c_1, \ldots, c_{k_1}$; the rows where they and p meet, $r_1, \ldots, r_{k_1}$; the columns of the elements ≥ 0 in these rows, $c_1, \ldots, c_{k_2}$; etc. If and when the row of α appears (Fig.3) a circuit (see 2.1) has been found, and hence a greater permutation sum. If not (Fig.4) we end up with a "tree" of k columns and k rows, $1 \leq k \leq n - 1$, such that the $k(n - k)$ elements in the k rows and remaining $n - k$ columns are negative; let β be a greatest of these elements. Subtract β from the k rows and add it to the k columns. If $\alpha + \beta \leq 0$ the number of positive elements outside p has decreased. If not the tree has grown. After a finite number of steps we are bound to find a greatest permutation sum. Its unicity can be similarly checked.

Retracing our steps we see that we have also found an $\epsilon L_0' \geq$ the given matrix and equal to it along a permutation with the greatest sum [1,2,14]. This requirement, however, does not in general uniquely determine the $\epsilon L_0'$.

To try to overcome the combinatorial difficulty of the problem we may continuize it, making not only the permutation sums $\Sigma_{ij} a_{ij} p_{ij}$ but also their weighted arithmetic means $\Sigma_{ij} a_{ij} \bar{p}_{ij}$, $\bar{p} \in P$, eligible, obviously without affecting the result. We now have a game [6] or linear maximization problem, and can use the fact that P has only n^2 faces. Any of the general descent procedures for linear programming [15] can be adapted to our case. From the point of view of the simplex method the assignment problem is still highly degenerate: every vertex belongs not to $(n - 1)^2$ but to $n^2 - n$ faces, and has not $(n-1)^2$ but $\sum_{k=2}^{n} n!/(n-k)!k \sim (n-1)!e$[1] adjacent (neighboring) vertices. The method has, though, been worked out for the, more general, transportation problem [16] and seems to be equivalent to the above tree algorithm. Note that adjacent vertices belong to permutations obtainable from each other by exchanging along a circuit [17] as we shall see in the next section.

4. The equidistribution problem. Several essential features of the assignment problem are preserved in the following generalization to a rectangular matrix $a = (a_{ij})$, $i = 1,\ldots,m$, $j = 1,\ldots,n$, $m \le n$.

We call a non-negative matrix $\bar{p} = (\bar{p}_{ij})$ an equidistribution if its rows have equal sums $\rho > 0$ and also its columns have equal sums $\sigma = \rho m/n$. We consider equidistributions with fixed ρ and σ. Several choices each have their merits: $\rho = 1/m$, $\sigma = 1/n$, $\sum_{ij} \bar{p}_{ij} = 1$; $\rho = 1$, $\sigma = m/n$, $\bar{p}$ can be completed by mutually equal elements to a doubly-stochastic matrix; $\rho = d/m$, $\sigma = d/n$, $\sum_{ij} \bar{p}_{ij} = d = (m,n)$ (the greatest common divisor of m and n) gives a also doubly-stochastic matrix for $m = n$; $\rho = n/d$, $\sigma = m/d$ does the same; $\rho = n$, $\sigma = m$.

To find (with given ρ and σ) an equidistribution with greatest $\sum_{ij} a_{ij}\bar{p}_{ij}$ is the equidistribution problem. We have, as before, an $(m-1)(n-1)$-space L_1 spanned by the equidistributions and defined by $\sum_i a_{ij} = \rho$, $\sum_j a_{ij} = \sigma$; its parallel L_0 through 0 is defined by $\sum_i a_{ij} = \sum_j a_{ij} = 0$. The orthogonal complement, namely the $(m+n-1)$-space L_0' consisting of all $(\lambda_i + \mu_j)$, meets L_1 at c, $c_{ij} = \rho/n = \sigma/m$, centroid of the equidistribution polyhedron P.

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1) The left divided by the right side attains (just as the entirely unrelated volume of the unit sphere in n-space) its greatest and next to greatest value for $n = 5$ and $n = 6$.

That c is the centroid of the vertices of P, of the edges of $P, \ldots,$ of P itself, is obvious since in each case the centroid has to be invariant under permutations of the rows as well as under permutations of the columns, and c is the only point on L_1 with this property. The study of the polyhedron P in the general case will throw some light on the square case $m = n$ as well.

The number of faces of P is mn, with two exceptions, since if one hyperplane $a_{ij} = 0$ were to contain no face, none would, and the inequalities are not identically fulfilled on L_1, except for $m = 1$, when $P = L_1$ is a point and has 0 faces; and since no two of the hyperplanes coincide on L_1, except for $m = n = 2$, when P is a segment and has 2 faces.

Except for $d = m$ the nonzero elements of a vertex of P are no longer all equal. A vertex of P can be obtained [8] by drawing a diagonal of the m by n rectangle enclosing a and setting $\bar{p}_{ij}$ proportional to the length of the segment of the diagonal in the i,j-square (Fig.5); all other vertices are found from this one by a permutation of the rows and a permutation of the columns. Alternatively divide a segment into m equal parts α_i (in some order) and into n equal parts β_j, and let $\bar{p}_{ij}$ equal the common part of α_i and β_j. It is easy to infer that the number of vertices is $m!\, n!/d!\, \ell^{n-\ell m+d}(\ell-1)!^m\, 2^{\epsilon d})$ where $\ell = [n/m]$ and $\epsilon = 0$ or 1 as $d =$ or $< m$. The elements of a vertex are $kd\rho/n = kd\sigma/m$, $k = 0,1,\ldots,m/d$.

The linear subspace defined by $a_{ij} = 0$ for all i and j except $1 \leq i-km/d \leq m/d$, $1 \leq j-kn/d \leq n/d$, $1 \leq k \leq d$, meets P in an $(m-d)(n-d)/d$ -dimensional <u>principal face</u> with $(m/d)!^d\, (n/d)!^d/(\ell^{n-\ell m+d}(\ell-1)!^m)$ vertices. Every vertex belongs to exactly one principal face. P has $m!n!/(m/d)!^d(n/d)!^d d!$ principal faces. For $d = m$ a principal face is a vertex, for $d = 1$ it is P.

The determination of all edges (segments connecting adjacent vertices) of P is simple for $m = n$. After a permutation of the rows and one of the columns, one of two given vertices may always be supposed to be the identity matrix, while the other consists of one or more circuits. The face hyperplanes common to both contain also every vertex defined by a subset of these circuits. For an edge, that is, for no additional vertex to arise thus, it is therefore sufficient and necessary that the number of proper circuits (circuits with more than one element) be exactly one. The total number of edges is $\sum_{k=2}^{n} n!^2/(n-k)!\,2k$.

If $d = m$ every vertex can be obtained from every other by a permutation of columns. If this permutation (after omitting ineffectual circuits) has more than one proper circuit again additional vertices belong to the smallest face containing the two vertices. If it has only one proper circuit, say $j_1 \cdots j_k$, and if in the first vertex the corresponding rows $i_1, \ldots, i_k$ (with $\bar{p}_{i_1 j_1} \neq 0$) are all different, it is easily seen that no additional vertices exist; on the other hand if, say, $i_1 = i_h$ then $j_1 \cdots j_k$ may be replaced by the two, not necessarily proper, circuits $j_1 j_{h+1} \cdots j_k$ and $j_2 \cdots j_h$. Hence two vertices are adjacent if and only if one can be obtained from the other by a permutation of the columns with a single proper circuit and such that the rows corresponding, via the nonzero elements of the vertex, to the columns of the circuit are all different. The number of edges from a vertex is $\sum_{k=2}^{m} m!\,n^k/(m-k)!\,m^k k$, altogether $\sum_{k=2}^{m} m!\,n!\,n^k/(n/m)!^m (m-k)!\,m^k 2k$.

4.1. Further complications enter into the description of all edges for $m < n$. We can give criteria for a subset of the hyperplanes $a_{ij} = 0$ to contain a face and to be its full defining set; but the application of these criteria and the determination of the dimension of the face are easy only in special cases, e.g., for small m and n.

To formulate the criteria first note that an equidistribution cannot be over-semireducible, that is, cannot have an "over half size grid" of zero elements $\bar{p}_{ij} = 0$, $i = i_1,\ldots,i_\mu$, $j = j_1,\ldots,j_\nu$, $\mu/m + \nu/n > 1$, for the other elements in the same rows would sum to $\mu\rho > (n-\nu)\sigma$. Secondly, if an equidistribution is semireducible, that is, has a "half size grid" of zeros $\bar{p}_{ij} = 0$, $i = i_1,\ldots,i_\mu$, $j = j_1,\ldots,j_\nu$, $\mu/m + \nu/n = 1$, then it is reducible, that is, has zeros also at $i \neq i_1,\ldots,i_\mu$, $j \neq j_1,\ldots,j_\nu$ (in the "complement"), for the elements with $i = i_1,\ldots,i_\mu$, $j \neq j_1,\ldots,j_\nu$ sum to $\mu\rho = (n - \nu)\sigma$; we say that an equidistribution cannot be properly semi-reducible.

Theorem 1. A set of zeros occurs in some equidistribution if and only if it contains no over half size grid.

Theorem 2. A set of zeros is the set of all zeros of some equidistribution if and only if it contains no over half size grid, and with every half size grid contains its complement.

Corollary. If $d = m$ then a set of less than mn zeros is the set of all zeros of some equidistribution if and only if, with every half size grid, it contains its complement.

The corollary follows from Theorem 2 by remarking that for $d = m$ an over half size grid contains a half size grid, and that by twice forming the complements of all half size grids contained one gets all mn elements.

To prove Theorem 1 (by double use of the duality or transposition theorem for linear inequalities [18]) we examine the relations $\bar{p}_{ij} \geq 0$, $\Sigma_j \bar{p}_{ij} = \rho$, $\Sigma_i \bar{p}_{ij} = \sigma$ for elements $\bar{p}_{ij}$ outside the given set of zeros. We homogenize to $\bar{p}_{ij} \geq 0$, $\Sigma_j \bar{p}_{ij} - \rho q = 0$, $\Sigma_i \bar{p}_{ij} - \sigma q = 0$, $q > 0$. By duality this is insolvable if and only if $\lambda_{ij} - \mu_i - \nu_j = 0$, $\rho\Sigma\mu_i + \sigma\Sigma\nu_j + \kappa = 0$, $\lambda_{ij} \geq 0$, $\kappa > 0$ is solvable; rewrite as $\mu_i + \nu_j \geq 0$, $\rho\Sigma\mu_i + \rho\Sigma\nu_j < 0$. For a solution we may, after rearranging rows and columns, suppose $\mu_1 \le \dots \le \mu_m$, $\nu_1 \le \dots \le \nu_n$. Then if $\mu_i + \nu_j \ge 0$ the same is true if i, or j, is increased. We can therefore alter our solution so that μ_i equals $-\lambda_1$ where λ_1 is the first ν_j relevant to this μ_i (a full row of zeros would be an over half size grid); that the next μ_i to which a smaller ν_j is relevant equals $-\lambda_2$ where $\lambda_2 < \lambda_1$ is the first ν_j relevant to this μ_i; etc. All μ_i between $-\lambda_1$ and $-\lambda_2$ can be set equal to $-\lambda_1$, etc. Now all ν_j after λ_1 can be set equal to λ_1, all those between λ_1 and λ_2 equal to λ_2, etc. We arrive at numbers $\lambda_1 > \dots > \lambda_k$ such that $(\mu_1, \dots, \mu_m) = (-\lambda_1\ (\alpha_1 \text{ times}), \dots, -\lambda_k\ (\alpha_k \text{ times}))$, $(\nu_n, \dots, \nu_1) = (\lambda_1\ (\beta_1 \text{ times}), \dots, \lambda_k\ (\beta_k \text{ times}))$, and $\gamma_1\lambda_1 + \dots + \gamma_k\lambda_k < 0$ where $\gamma_1 = \sigma\beta_1 - \rho\alpha_1$, etc., whence $\gamma_1 + \dots + \gamma_k = 0$. The system $-(\gamma_1\lambda_1 + \dots + \gamma_k\lambda_k) > 0$, $\lambda_1 - \lambda_2 > 0$, $\dots$, $\lambda_{k-1} - \lambda_k > 0$ is, by duality, solvable if and only if $-\gamma_1\tau + \tau_1 = 0$, $-\gamma_2\tau - \tau_1 + \tau_2 = 0$, $-\gamma_3\tau - \tau_2 + \tau_3 = 0$, $\dots$, $-\gamma_k\tau - \tau_{k-1} = 0$, $(\tau, \tau_1, \dots, \tau_{k-1}) \ge (0, \dots, 0)$ is insolvable. We may set $\tau = 1$, $\tau_1 = \gamma_1$, $\tau_2 = \gamma_1 + \gamma_2$, $\dots$, $\tau_{k-1} = \gamma_1 + \dots + \gamma_{k-1}$. This means that some $\gamma_1 + \dots + \gamma_i < 0$; but then the first $\alpha_1 + \dots + \alpha_i$ rows and $n - (\beta_1 + \dots + \beta_i)$ columns intersect in an over half size grid. Q.E.D.

If even no half size grid occurs in the given set then $\tau_i = \gamma_1 + \dots + \gamma_i > 0$ so that $-(\gamma_1\lambda_1 + \dots + \gamma_k\lambda_k) \ge 0$, $(\lambda_1 - \lambda_2, \dots, \lambda_{k-1} - \lambda_k) \ge (0, \dots, 0)$ is insolvable. Hence $\mu_i + \nu_j \ge 0$, $\rho \sum \mu_i + \sigma \sum \nu_j \le 0$ has no solution except $\mu_1 = \dots = \mu_m$, $\nu_1 = \dots = \nu_n$ (if, say, the ν_j are unequal proceed as above; they remain unequal since $k = 1$ implies their equality even before altering). Thus $\lambda_{ij} - \mu_i - \nu_j = 0$, $\rho \sum \mu_i + \sigma \sum \nu_j + k = 0$, $(\lambda_{ij}, k) \ge (0, \dots, 0)$ is insolvable and there exists a $\bar{p} \in P$ with $\bar{p}_{ij} > 0$ outside the given set. To complete the proof of Theorem 2 note that if the set contains two complementary half size grids then the problem reduces to the construction of equidistributions on two disjoint grids on which the suppositions are again fulfilled.

4.2. As an example consider the case $m = 2$. Here an over half size grid is either a column or more than half of a row. The faces of P are therefore either of type F_j: $a_{11} = \dots = a_{1j} = 0$, $j < n/2$, or $F_{jj'}$: $a_{11} = \dots = a_{1j} = 0$, $a_{2n} = \dots = a_{2,n-j'+1} = 0$, $j' \leq j < n/2$, or F: $a_{11} = \dots = a_{1,n/2} = 0$, $a_{2,n/2+1} = \dots = a_{2n} = 0$. Their dimensions are $n - j - 1$, $n - j - j' - 1$, 0. The number f_δ of δ-dimensional faces is thus $\binom{n}{\delta+1} \sum_{k=[n/2-\delta]}^{[(n-1)/2]} \binom{n-\delta-1}{k}$ (for $\delta > n/2 - 1$ this is $2^{n-\delta-1}\binom{n}{\delta+1}$), but $f_0 = \binom{n}{n/2}$ for even n. Setting $f_{-1} = 1$ we obtain the following values for $n \leq 10$, where a check is provided by $\sum -^{\delta} f_\delta = 0$. $\sum f_\delta \sim 3^n$.

n \ δ	-1	0	1	2	3	4	5	6	7	8	9
1	1	1									
2	1	2	1								
3	1	6	6	1							
4	1	6	12	8	1						
5	1	30	60	40	10	1					
6	1	20	90	120	60	12	1				
7	1	140	420	490	280	84	14	1			
8	1	70	560	1120	980	448	112	16	1		
9	1	630	2520	4200	3780	2016	672	144	18	1	
10	1	252	3150	8400	10500	7560	3360	960	180	20	1

To find the volume v of P project on $a_{21} = \dots = a_{2n} = 0$; the projection cosine is $\gamma =$ vol A/vol(A,A) where $A = \begin{pmatrix} 1 & & 0 & -1 \\ & \ddots & & \vdots \\ 0 & & 1 & -1 \end{pmatrix}$ ($n - 1$ by n). Since $(\text{vol } A)^2 = |AA'| = \begin{vmatrix} 2 & \ddots & 1 \\ 1 & \ddots & 2 \end{vmatrix} = 2n - 3$ and $(\text{vol}(A,A))^2 = \begin{vmatrix} 4 & \ddots & 2 \\ 2 & \ddots & 4 \end{vmatrix} = (2n-3)\, 2^{n-1}$, we have $\gamma = 2^{(1-n)/2}$. Denoting by $v_\tau = \tau^n n^{1/2}/n!$ the volume of the simplex

$a_{11} \geq 0, \ldots, a_{1n} \geq 0$, $\Sigma a_{1j} = \tau$, a familiar argument shows that $v\gamma = v_\rho - nv_{\rho-\sigma} + \binom{n}{2} v_{\rho-2} - + \ldots ([(n+1)/2]$ terms); e.g., for $\sigma = 1$ we obtain $v = 2^{(n-1)/2} n^{1/2} n!^{-1} ((n/2)^{n-1} - \binom{n}{1} (n/2-1)^{n-1} + \binom{n}{2} (n/2-2)^{n-1} - + \ldots)$. The projected P is the middle section between opposite vertices of an n-dimensional cube; for a unit cube ($\sigma=1$) the volume $v\gamma$ is $\sqrt{2}$, $3/4\sqrt{3}$, $4/3$, $115/192\sqrt{5}$, $11/20\sqrt{6}$ for $n = 2, \ldots, 6$. Quite similar but more cumbersome computations will give v for $m > 2$. A general expression, even for $m = n$, is still outstanding; for $m = n = 3$, $v = 9/8$.

Also the values of f_δ are harder to obtain for the $m = n$ sequence of polyhedra. They start:

n \ δ	-1	0	1	2	3	4	5	6	7	8	9
1	1	1									
2	1	2	1								
3	1	6	15	18	9	1					
4	1	24	240	978	1968	2176	1392	528	120	16	1

In every case, f_δ as a function of δ has one of the usual bell shapes (which seem to be most lopsided, with a peak at 1/3 the range, for the cube and its dual).

5. <u>Related problems</u>. We mention briefly various related problems and generalizations.

1. The transportation problem: to maximize $\Sigma_{ij} a_{ij} \bar{p}_{ij}$ for $\bar{p}_{ij} > 0$, $\Sigma_j \bar{p}_{ij} = \rho_i$, $\Sigma_i \bar{p}_{ij} = \sigma_j$ [8, 16].
2. The stair problem [19]: to find a greatest $\Sigma a_{i_k j_k}$ with first term a_{11} and last term a_{mn} such that every $(i_{k+1} - i_k, j_{k+1} - j_k)$ is $(1,0)$ or $(0,1)$.

3. The problem to find a convex curve, in the sense of 1.42, with greatest sum $\sum a_{ij}$.

4. In a rectangular matrix, to find a greatest sum $\sum a_{ij}$ with one term in every row and at most one term in every column.

5. Or else, with one term in every column and at least one term in every row.

6. Generalizations of the assignment and the equidistribution problem and of the above five problems to matrices with more than two indices, the candidate sums being taken with one, or with more, indices [20].

7. Given a matrix q_{jk} of zeros and ones to find, for given real numbers a_j, a greatest sum $\sum a_j q_{jk}$[21].

8. Or else, for positive a_j, to maximize $\sum a_j p_j$, $p_j > 0$, $\sum p_j q_{jk} \leq 1$, or equivalently to minimize $\max_k \sum \lambda_j q_{jk}/a_j$ (generalizing [6] to an arbitrary game matrix whose nonzero elements, in every row, are equal).

Of these, 4 and 5 share with the assignment and equidistribution problems the property of invariance under all permutations of rows and of columns. The corresponding polyhedra, and those pertaining to 1, 2 and 6, have (for large matrices) much fewer (highest-dimensional) faces than vertices; this suggests that "face approach" descent methods may work well in these cases.

In case 4, for $n > m$, the $n!/m!$ vertices span an $(mn - m)$-dimensional linear subspace, and a polyhedron whose $mn + m$ faces are in the hyperplanes $a_{ij} \geq 0$, $\sum_j a_{ij} \leq 1$. For positive matrices 4 belongs also to the polyhedron $a_{ij} \geq 0$, $\sum_j a_{ij} \leq 1$, $\sum_i a_{ij} \leq 1$, which arises from a special case of 8. Its vertices are matrices of ones and zeros, with at most one 1 in every row and column. In fact, suppose μ rows and ν columns of a vertex contain elements other than ones and zeros, and let, e.g., $\nu \geq \mu$. Suppose further that $\lambda \geq 0$ of these columns contain exactly one such element. Then the (μ,ν)-grid concerned contains at least $2\nu - \lambda$ nonzeros. On the other hand a vertex fulfills some mn of the inequalities as independent equations. Besides the equations $a_{ij} = 0$, any 1 in a vertex gives rise to one equation only, the other being dependent on it and the $a_{ij} = 0$. The (μ,ν)-grid gives rise to $\kappa \leq \mu + \nu - \lambda$ equations other than $a_{ij} = 0$. Thus $\mu + \nu - \lambda \geq 2\nu - \lambda$ whence $\mu = \nu$, $\kappa = 2\nu - \lambda$. Since the ν rows now sum to ν, so do the ν columns whence $\lambda = 0$. But the κ equations, together with the $a_{ij} = 0$ outside the grid, are not independent, a contradiction. - A third polyhedron equivalent for positive matrices to the former two is $\sum_j |a_{ij}| \leq 1$, $\sum_i |a_{ij}| \leq 1$, with $2^m n!/m!$ vertices.

To determine the faces and count the vertices in case 5, let us relax the second condition to "at least one term in λ given rows". Necessarily $\lambda \leq n$, and for $\lambda = n$ we have the assignment problem. For $\lambda < n$ consider, in the $(mn - n)$-dimensional linear subspace $\sum_i a_{ij} = 1$, the polyhedron defined by the $mn + \lambda$ inequalities $a_{ij} \geq 0$, $\sum_j a_{ij} \geq 1$. At a vertex at least $mn - n$ become equations, whence less than $2n$ do not, and we see that not every column contains more than one nonzero. But if we have a column with only one nonzero we fall back to the same question for $n - 1$. Hence a vertex consists only of zeros and ones, and the polyhedron is the one we were looking for in the first place. The recursion to $n - 1$ gives, for the number $\nu(m,n,\lambda)$ of vertices, the formula $\nu(m,n,\lambda) = (m - \lambda)\, \nu(m,n-1,\lambda) + \lambda\, \nu(m, n-1, \lambda-1)$. It is easy to verify the same (and the vanishing for $n = 0$) for $m^n - \binom{\lambda}{1}(m-1)^n + \binom{\lambda}{2}(m-2)^n - + \dots -^{\lambda}(m-\lambda)^n = \nu(m,n,\lambda)$. In particular, $\nu(m,n,n) = n!$ and (the vertex number of case 5) $\nu(m,n,m) = m^n - m(m-1)^n + \binom{m}{2}(m-2)^n - + \dots -^{m} 0^n$.

Deviating from the assignment problem in direction 7, we can restrict the set of admissible permutations. We mention only sets invariant if the rows and columns undergo the same permutation. Every permutation is thought of as the product of disjoint circuits of $\alpha_1 \geq \ldots \geq \alpha_k$ elements, $\Sigma\alpha_i = n$.

A. The $n!/2$ even permutations (those with even $n - k$).

B. The involutory (or symmetric) permutations (those with $\alpha_1 \leq 2$).

C. The fixpointless permutations (those with $\alpha_k \geq 2$).

D. The cycles (permutations with $\alpha_2 = 1$ or $k = 1$).

E. The intersections of these sets, like BD (transpositions) and CD (full cycles, [22]).

The corresponding polyhedra are apt to have many more faces than vertices. For A, the f_δ-table starts

n \ δ	-1	0	1	2	3	4	5	6	7	8	9
2	1	1									
3	1	3	3	1							
4	1	12	66	220	492	768	840	624	288	64	1

The last polyhedron consists of three 4-simplices in three orthogonal 3-spaces which span 9-space. The faces are 64 9-simplices. Every two vertices are adjacent.

Polyhedra with relatively few faces belong to BD (a simplex) and B. The latter consists of all symmetric doubly-stochastic matrices.

Finally it seems worth while to draw attention (besides to related non-linear problems [14, 23]) to sequences of doubly-stochastic matrices whose limits for $n \to \infty$, if piecewise continuous, might help determine the one-to-one piecewise continuous functions f maximizing $\int F(x,y)\, df$ with given kernel F defined in the unit square, and to a similar variation-theoretic application of problem 4.

Geometrical concepts used.

1. Let d, the dimension, be a positive integer. A row of d real numbers (coordinates) is called a point or vector in real d-dimensional space. The 0 row is called origin. Two vectors with vanishing sum of products of corresponding coordinates are orthogonal.
2. For a set S of points, ϵS means any element (point) v of S. The linear combinations $\sum \lambda_i v_i$ $(\sum \lambda_i = 1)$ of finitely many points of S form the linear subspace L spanned by S. If L can be spanned by $\delta + 1$ points but not by less, L and S are of dimension δ. L can be defined by $d - \delta$ linear equations. For $\delta = d - 1$, L is called hyperplane. A coordinate subspace is defined by the vanishing of $d - \delta$ coordinates.
3. Adding one and the same vector to every point of L gives a parallel subspace. Two linear subspaces are orthogonal if their parallels L_1 and L_2 through the origin are such that any ϵL_1 and any ϵL_2 are orthogonal. If together they span the whole space they are orthogonal complements.
4. For a finite set $S = (v_1, \ldots, v_m)$, $\sum v_i/m$ is the centroid or arithmetic mean; $\sum \lambda_i v_i$ $(\lambda_i \geq 0, \sum \lambda_i = 1)$ is a weighted arithmetic mean. The weighted arithmetic means of S form a continuum called the convex hull $[S]$ of S; $[S]$ is a convex polyhedron. $[S]$ can be defined by finitely many linear inequalities.
5. If the points of a convex polyhedron of dimension δ fulfill a linear inequality, those fulfilling the corresponding equation form a face (sometimes "face" stands for "$(\delta-1)$-dimensional face"). A zero-dimensional face is called vertex, a one-dimensional face edge; an edge connects two adjacent or neighboring vertices.
6. For $m = \delta + 1$, $[S]$ is a simplex, for $\delta = 1$ a segment. The points all of whose coordinates are $\geq a$ and $\leq b$ form a cube (unit cube for $b-a = 1$); two vertices of a cube are opposite if no face contains both.

7. For a simplex $[v_0,\ldots,v_d]$, the volume is $|\mathrm{Det}\,(v_1-v_0, v_2-v_0,\ldots,v_d-v_0)|/d!$; similarly for a simplex spanning a coordinate subspace. The volume of a polyhedron spanning such a subspace is obtained by decomposition into simplices of its own dimension the common part of any two of which has smaller dimension. For a polyhedron $[S]$ spanning an arbitrary linear subspace L of dimension δ, project on every coordinate subspace $L_j, j = 1,\ldots, \binom{d}{\delta}$, of dimension δ (by replacing $d - \delta$ coordinates by 0); then $v^2 = \Sigma v_j^{\,2}$, where v_j are the volumes of the projections and $v \geq 0$ the volume of $[S]$, and the projection cosine v_j/v depends only on L and L_j.

Figures

Fig. 1

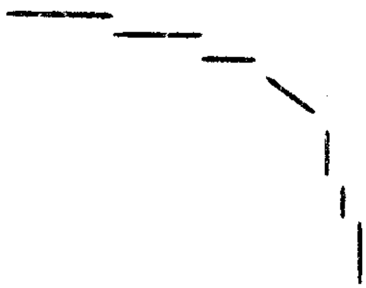
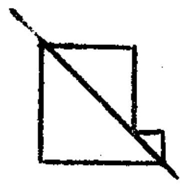


Fig. 2



(in figures 2, 3, 4, for the sake of clarity, the permutation sum under consideration is the main diagonal.)

Fig. 3

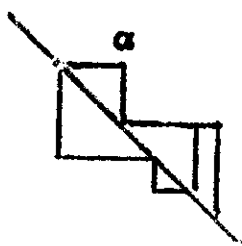


Fig. 4

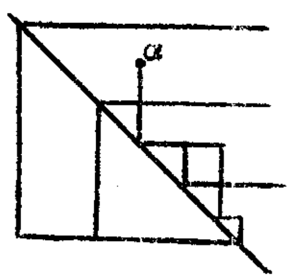
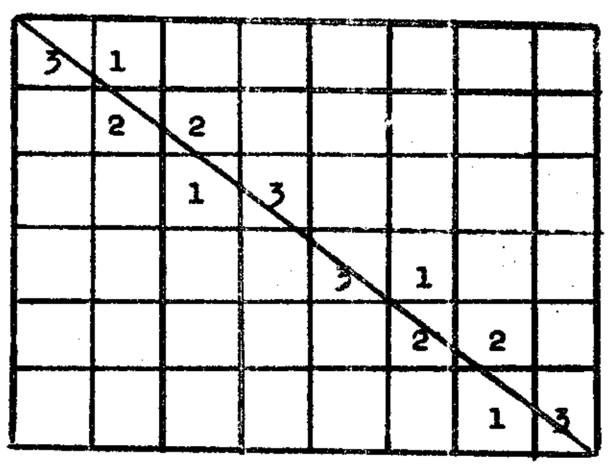


Fig. 5



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