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Separation Theorems for Convex Sets[1/]

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During the last decade separation theorems for convex sets have emerged as a basic mathematical tool in economics (Theory of Games, Activity Analysis [8], Welfare Economics [1],[3],[4],[10]. The purpose of this paper is to gather, for convenience, what seem to be the most important of these results and their simplest proofs. Two corollaries have been given for each theorem; many more detailed results could be derived from them in the same immediate fashion. The paper is entirely expository: all results and techniques of proof are standard in mathematics.

All sets considered are subsets of a finite Euclidean space R^n.

A hyperplane is bounding for a set A if A is contained in one of the closed half spaces determined by the hyperplane.

Theorem 1. Let $\bar{A}$ be a closed, convex set. If $x^o \notin \bar{A}$

1) there is a point x' of $\bar{A}$ closest to x^o.

2) the hyperplane H through x', perpendicular to $x^o x'$, is bounding for $\bar{A}$.

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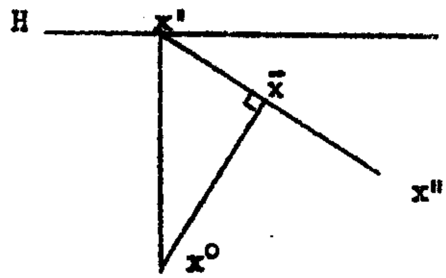
1) The distance $d(x^o, x)$ is a <u>continuous</u> function of x on $\bar{A}$ which is <u>closed</u>. Moreover only a <u>bounded</u> part of $\bar{A}$ is relevant. Therefore $d(x^o, x)$ reaches its minimum at x'.

2) Suppose that there is a point x'' of $\bar{A}$ strictly on the same side of H as x^o. Draw the figure in the plane x^o, x', x''. The point $\bar{x}$ of $x'\,x''$ such that $x^o\,\bar{x}$ is normal to $x'\,x''$ would be in $\bar{A}$ and closer to x^o than x'.



<u>Corollary 1.</u> <u>Let A be a convex set, its closure $\bar{A}$ is the intersection I of the closed half spaces containing A.</u>

A closed half space containing A also contains $\bar{A}$, so $\bar{A} \subset I$. Moreover, by theorem 1, if $x^o \notin \bar{A}$, $x^o \notin I$.

Let C be a cone with vertex 0, its polar C* is the set $\left\{ y \,\middle|\, y.\,x \leq 0 \text{ for all } x \in C \right\}$. C* is clearly a cone with vertex 0 which is closed and convex.

<u>Corollary 2.</u> <u>If C is a closed, convex cone with vertex 0, $(C^*)^* = C$.</u>

From the definition of the polar, $C^{**} \supset C$. Moreover if $x^o \notin C$, $x^o \notin C^{**}$. To see this, consider the hyperplane H introduced in theorem 1.

If $x' = 0$, trivially H contains 0. If $x' \neq 0$, $x^o\,x'$ is normal to the half-line Ox' which is thus contained in H, so that in any case H contains 0. Therefore there is a normal y to H at 0 such that $y\,.\,x \leq 0$ for all $x \in C$, and $y.\,x^o > 0$. Thus $y \in C^*$ and $x^o \notin C^{**}$.

If $x_0, \ldots, x_n$ are n + 1 points lying in no hyperplane, the set of all their convex combinations is called the n-simplex determined by these points.

Lemma Let A be a convex set, its interior is equal to the interior of its closure $\bar{A}$.

Obviously Int $A \subset$ Int $\bar{A}$. Let then y be an interior point of $\bar{A}$, there is an n-simplex $(x_0, \ldots, x_n)$ of points of $\bar{A}$ to which y is interior. There are n + 1 points of A $x'_0, \ldots, x'_n$ generating an n-simplex, close enough to the corresponding $x_0, \ldots, x_n$ so that y is an interior point of the n-simplex $(x'_0, \ldots, x'_n)$ and therefore of A.

Theorem 2. Let A be a convex set. If $x^o \notin A$, there is a hyperplane through x^o, bounding for A.

Denote by $\bar{A}$ the closure of A. If $x^o \notin \bar{A}$ one takes the hyperplane through x^o parallel to H introduced in theorem 1.

If $x^o \in \bar{A}$, it is not an interior point of $\bar{A}$ (since in that case it would also be an interior point of A by the lemma, contradicting $x^o \notin A$). Thus x^o is the limit of a sequence (x^k) where $x^k \notin \bar{A}$. For every k there is a hyperplane through x^k of equation $p^k . x = p^k . x^k$ bounding for $\bar{A}$. We can choose p^k in such a way that $p^k . x \geq p^k . x^k$ for all $x \in \bar{A}$, and $|p^k| = 1$.

Extract then a converging subsequence $p^{k'} \longrightarrow p^o$. Clearly $p^o . x \geq p^o . x^o$ for all $x \in \bar{A}$, and $|p^o| = 1$. The hyperplane of equation $p^o . x = p^o . x^o$ satisfies all the requirements. The mathematical content of the basic theorem of the New Welfare Economics is theorem 2 (See Debreu [4]).

A hyperplane H is separating for two sets A,B if A is contained in one of the closed half-spaces determined by H, B in the other.

Corollary 3. Let A, B be two disjoint, convex sets, there is a hyperplane separating them.

Consider C = A - B, the set of all x-y where $x \in A$, $y \in B$. C is easily seen to be convex. $0 \notin C$ since A, B are disjoint. Thus there is a hyperplane $p.z = 0$ through 0, bounding for C, i.e., $p. (x - y) \geq 0$ for all $x \in A$, $y \in B$; that is

$p.x \geq p.y$. Take $\alpha = \operatorname{Inf}_{x \in A} p.x$; the hyperplane $p.z = \alpha$ clearly separates A and B.[2/]

On this corollary are based simple proofs of the two-person zero-sum game theorem and the proofs of the New Welfare Economics theorem by Arrow [1], Debreu [3].

Let Ω be the non-negative orthant (set of points with non-negative coordinates), $\mathring{\Omega}$ its interior.

Corollary 4. Let C be a closed, convex cone with vertex 0. If $C \cap \Omega = 0$, then $C^* \cap \mathring{\Omega} \neq \emptyset$.

If $C^* \cap \mathring{\Omega} = \emptyset$, C^* and $\mathring{\Omega}$ are separated by a hyperplane (by theorem 2) which also separates C^* and Ω. It therefore goes through 0 and has equation $p.x = 0$. So (1) for all $x \in C^*$, $p.x \leq 0$; (2) for all $x \in \Omega$, $p.x \geq 0$. From (1) $p \in C^{**}$, i.e., $p \in C$ (Corollary 2); from (2) $p \in \Omega$. This contradicts $C \cap \Omega = 0$.

This corollary is the basic mathematical tool of Activity Analysis (Koopmans [8] Chap. 3).

A key to a bibliography on convex sets will be found in the three papers by Klee [5], [6], [7]; the volume by Bourbaki [2], the monograph [8], and the mimeographed notes [9] may be added.

2/ The relative interior of a convex set S is its interior with respect to the smallest linear space containing S. As an example of application of Corollary 3 one has: Let X, Y be two convex sets whose relative interiors are disjoint, there is a hyperplane separating them. Call A, B the relative interiors of X, Y. A, B are convex, disjoint; there is a hyperplane H separating them. H also separates $\bar{A}$ and $\bar{B}$. Since $X \subset \bar{A}$, $Y \subset \bar{B}$, the statement is proved.

References

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