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An Index to Some Banach Space Definitions and Results

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In all that follows K will be the field of real or complex numbers.

Definition 1. A <u>linear space</u> V over K is a set with a binary operation + (defined for all $x, y \in V$ so that $x + y \in V$) and a "scalar multiplication" αx, defined for $\alpha \in K$, $x \in V$ so that αx is again in V and where

(1) $x + y = y + x$

(2) $x + (y + z) = (x + y) + z$

(3) $x + z = y + z \longrightarrow x = y$

(4) $\alpha(x + y) = \alpha x + \alpha y$

(5) $(\alpha + \beta)(x) = \alpha x + \beta x$

(6) $\alpha(\beta x) = (\alpha\beta) x$

(7) $1 x = x$

for all $x, y, z \in V$, $\alpha, \beta \in K$.

It follows that there is an element $0 \in V$ so that $x + 0 = x$ for all $x \in V$, and if $x \in V$ there exists an element which we denote by $-x$ so that $x + (-x) = 0$.

Definition 2. A <u>normed linear space</u> E over K is a linear space over K on which is defined a real valued function $\| \; \|$ (called a norm) so that

(1) $\|x\| \geqslant 0$; $\quad \|x\| = 0 \longleftrightarrow x = 0$

(2) $\|\alpha x\| = \|\alpha\| \; \|x\|$

(3) $\|x + y\| \leqslant \|x\| + \|y\|$

for all $x, y \in E$, $\alpha \in K$.

Note: if we define $\rho(x, y) = \| x - y \|$ for $x, y \in E$ we obtain a metric on E. Whenever we will speak of convergence in norm on E we will mean convergence in this metric.

Definition 3. A metric space M with metric ρ is said to be complete in ρ if: given any sequence $\{x_n\}$, $x_n \in M$ so that for any preassigned $\epsilon > 0$ there exists an integer N_ϵ so that if $n, m > N_\epsilon$ then $\rho(x_n, x_m) < \epsilon$, then this sequence converges to a point x which is in M.

Such a sequence will be termed a Cauchy sequence. A space is complete if every Cauchy sequence in it has a limit in it.

Definition 4. A normed linear space is said to be a Banach space if it is complete in the metric defined by the norm.

Note: Henceforth all (or nearly all) the theorems will refer to Banach spaces; it is true that many of these hold even in the absence of completeness and sometimes even in more general situations, but for the sake of uniformity we assume completeness.

Definition 5. Let E, F be two Banach spaces. Suppose $T: E \rightarrow F$. T is called a linear function (or transformation) if
$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$ for all $x, y \in E$ and $\lambda, \mu \in K$.

Definition 6. The unit sphere of E is $\{x \in E \mid \|x\| \leq 1\}$.

Lemma A. A linear function from E to F is continuous if and only if it is bounded on the unit sphere of E, i.e., if and only if $\|T(x)\| \leq M$ for all x with $\|x\| \leq 1$. (The first norm the one in F, the second one, that in E.)

Definition 7. A linear functional on E is a linear function mapping E into K, the real or complex numbers.

Definition 8. If E is a Banach space, then E^*, the conjugate or dual space of E, is the set of all continuous linear functionals defined on E. (Note: as yet we have no guarantee that E^* has anything other than the 0-functional.)

Definition 9. If $f \in E^*$ then $\|f\| = \sup_{\substack{\|x\| \leq 1 \\ x \in E}} |f(x)|$

Theorem B. E^* forms a Banach space under the norm defined in definition 9.

Note: If E is finite-dimensional then $E = E^*$.

Theorem C. (Hahn-Banach.) Let E be a Banach space, and V a subspace of E. Any continuous linear functional on V can be extended to one on E in a norm preserving fashion.

Corollary 1. If $x_0 \in E$, there exists an $f \in E^*$ with $\|f\| = 1$ and $|f(x_0)| = \|x_0\|$.

(Thus E^* is never trivial.)

Corollary 2. If $x \neq y \in E$ there exists on $f \in E^*$ with $f(x) \neq f(y)$.

Note: if $f \in E^*$, $x \in E$ we can define $\tilde{x}(f) = f(x)$. $\tilde{x}$ defines a linear functional on E^*. By the H-B theorem $\tilde{x}$ is continuous. If $x \neq y$ then $\tilde{x} \neq \tilde{y}$ (by H-B). So if $E^{**} = (E^*)^*$ we have a natural imbedding on E into E^{**}; i.e. $E^{**} \supset E$ (actually this imbedding, by H-B, preserves norm). So:

Lemma D. $E^{**} \supset E$

Definition 10. E is said to be reflexive if $E = E^{**}$.

Weak Topology

Definition 11. If $\{x_i\} \in E$ then

$x_i \longrightarrow x$ weakly if for all $f \in E^*$, $f(x_i) \longrightarrow f(x)$.

Lemma E. If $x_i \longrightarrow x$ in norm then $x_i \longrightarrow x$ weakly. (Converse false).

Note: This is not sufficient to define a topology.

Definition 12. The weak topology on E is the weakest topology (i.e. one with fewest number of open sets) on E preserving the continuity of the elements of E^*.

Formal Definition:

We define the neighborhoods of 0 in E to be: take any finite number of elements $f_1, \ldots, f_n \in E^*$, any $\epsilon > 0$, let $U = \{x \in E \mid |f_i(x)| < \epsilon \text{ for } i=1,\ldots,n\}$; these U's form our system of neighborhoods of 0 in the weak topology.

Weak *- topology.

This is a topology defined on E^* in the same way we introduced the weak topology on E, but using not all of E^{**} but merely the subset $E \subset E^{**}$; i.e. to get neighborhoods take any finite number of elements $x_1, \ldots, x_n \in E$, $\epsilon > 0$ etc., etc.

Theorem F. (Aloaglu) The unit sphere S^* of E^* is compact in the weak *-topology.

Theorem G. Let E be any Banach space, S^* the unit sphere of E^*. Then E can be imbedded, in a norm-preserving fashion into $C(S^*)$ the set on continuous functions on S^* to K. (Thus we have a canonical representation of any Banach space as a set of continuous functions from a compact, Hausdorff space to K).

Theorem H. A Banach space is reflexive if and only if its unit sphere is weakly compact.

Definition 13. If X is any topological space, then $S \subset X$ is nowhere dense if the interior of $\bar{S}$ (the closure of S) is void.

Definition 14. T is said to be of the <u>first category</u> if it is the union of a countable number of nowhere dense sets. Otherwise it is of the <u>second category</u>.

Theorem I. A Banach space is of the second category.

Theorem J. (Uniform Boundedness Theorem). Let E, F be Banach spaces, U a set of linear functions from E into F. If for all $T \in U$

$\|T(x)\| \leq M(x)$, $x \in E$ (M independent of T) then

$\|T\| < A$ (a fixed number) for all $T \in U$.

Theorem K. (Closed Graph Theorem).

Let E, F be Banach spaces. T a linear function from E into F. Suppose the set of all $(x, T(x))$ is closed in $E \times F$ (i.e. the graph of T is closed). Then T is continuous.

Definition 15. Given E, E^* we define for $x \in E$, $y \in E^*$, $(x, y) = y(x)$.

Suppose $T: E \longrightarrow F$ (linear).

We define

Definition 16. $T^*: F^* \longrightarrow E^*$ by

$(x\,T, y) = (x, y\,T^*)$ [$x\,T = T(x)$, notation]

$x \in E$, $y \in F^*$.

<u>Note</u>: T^* might not exist. <u>If it does it is unique</u>.

Theorem L. T^* exists if and only if T is continuous.

<u>Note</u>: If E and F are finite dimensional then T is a matrix and T^* is exactly the usual transpose, complex-conjugate of this matrix.

Theorem M. $T^{**} = T$

$\|T^*\| = \|T\|$

$(TU)^* = U^* T^*$ (here we need 3 spaces).

Definition 17. $K \subset E^*$ is <u>regularly convex</u> if for every $g \notin K$ there exists an $x_0 \in E$ so that $\sup_{f \in K} f(x_0) < g(x_0)$

<u>Note</u>: regular convexity $\longrightarrow$ convexity in usual sense.

Theorem N. (Krein-Milman). Let $K \subset E^*$ be a bounded regularly convex set; then the set of extreme points of K is not empty and its closed convex hull is exactly K.

Theorem O. (Krein-Milman; Kelley generalization).

Let C be a compact convex subset of a Banach space (actually, merely a locally convex space). Then C has extreme points and is their convex hull.

Definition 18. A linear space V over K is said to be an <u>inner-product</u> space if for all x, y $\in$ V we obtain an element (x, y) <u>in K</u> so that

(1) $(x, x) \geq 0$; $(x, x) = 0 \longleftrightarrow x = 0$

(2) $(x, y) = \overline{(y, x)}$ (complex conjugate)

(3) $(\alpha x + \beta z, y) = \alpha(x, y) + \beta(z, y)$.

<u>Note</u>: if we define, for V an inner product space, $\|x\| = \sqrt{(x, x)}$ this defines a norm on V.

Thus an inner product space becomes a normed linear space.

Definition 19. An inner product space H is called a <u>Hilbert space</u> if H is complete in the norm defined by $\|x\| = \sqrt{(x, x)}$.

So every Hilbert space is a Banach space; converse is false.

<u>Note</u>: if V is an inner product space and $\|x\| = \sqrt{(x, x)}$ then for all x, y $\in$ V

$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ (parallelogram law).

Theorem P. A Banach space is a Hilbert space if and only if the parallelogram law holds true for its norm.