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Lagrangian Saddle Points in Banach Spaces (Summary of results)

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In what follows I present some results on the existence of saddle points of the type known from the theory of (linear and non-linear) programming.

These results have not as yet been scrutinized with the required rigor; it is not at all unlikely that imperfections will be discovered. On the one hand, the listing of conditions for the validity of the theorems may be incomplete or inaccurate; on the other, some of the conditions imposed are unnecessarily restrictive, as for instance in the case of the reduction of vectorial to scalar extremization problems. Most propositions not involving differentiation can be extended to a class of very general linear topological spaces. In fact, a further extension to topological additive groups is in prospect.

1. Terminology and notation. Unless otherwise stated, we shall conform to the principles given in CCDP 415. (The basic literature references are also given there.)

If K is a convex cone in a Banach space W, we write $w' \geqq w''$ to mean $(w' - w'') \in K$; thus if 0 denotes the identity element of addition in W, $w \geqq 0$ means $w \in K$. K is said to be pointed if $w \in K$, $-w \in K$ imply $w = 0$. An additive continuous (hence also homogeneous and bounded) functional w^* on W is said to be (a) non-negative (written: $w^* \geqq 0$), (b) positive, and (c) strictly positive on a (not necessarily pointed !) cone K if $w \in K$ implies respectively (a) $w^*(w) \geqq 0$; (b) $w^*(w) \geqq 0$ and, for some $w_o \in K$, $w^*(w_o) > 0$; (c) $w^*(w) \geqq 0$ and, $w^*(w) > 0$ if $-w \notin K$.

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1. This paper discusses mathematical tools for CCDP Economics No. 2070, "Decentralized Resource Allocations", by L. Hurwicz.

A function f(x) with a domain in X and range in Y is said to be concave if and only if

$$f(\theta x' + (1-\theta)x'') \geqq \theta f(x') + (1-\theta) f(x'')$$

for all $x', x'' \in X$ and $0 \leq \theta \leq 1$.

2. Reduction of vectorial to scalar extremization problems

2.1 Theorem 1. Let X, Y, Z be Banach spaces, Y separable; f and g concave continuous functions from X to Y and Z respectively; X^+, Y^+, and Z^+ closed convex cones in X, Y, Z respectively, Y^+ pointed; the set $X^c \equiv X^+ \cap g^{-1}(Z^+)$ non-empty and compact.

Let $\bar{x} \in X$ be maximal in the sense that (1) $\bar{x} \in X^c$, and (2) $x \in X^c$, $f(x) \geqq f(\bar{x})$ imply $f(x) = f(\bar{x})$.

Then there exists an additive continuous functional y* which is strictly positive on Y^+ and has the property that $x \in X^c$ implies

$$y^*[f(x)] \leqq y^*[f(\bar{x})].$$

2.2 This problem arises when one wishes to "maximize vectorially" the vector-valued function f(x), subject to the conditions $x \geqq 0$ and $g(x) \geqq 0$; problems of this type were considered (for finite-dimensional spaces) by Kuhn and Tucker, /4/, and Slater, /5/, and, of course, in activity analysis (linear programming) literature for f, g linear. (The interpretation: X - the "Activity level" space, Y and Z the "desired" and "primary" commodity spaces respectively.) The approach adopted in the present note (following /5/) consists in reducing the "vectorial" maximization problem involving f(x) to one of "scalar" maximization of y* [f(x)] as a function of x. The reasons for insisting on the strict positiveness of y* are mentioned in 2.4.

It may be noted that $f(\bar{x})$ is an efficient point in Koopmans' sense. Theorem 1 plagiarizes Slater's Lemma 2./5/; but while Slater's conditions are in some respects less restrictive, he only proves the existence of a positive (not necessarily strictly positive) y^*. This is related to the fact that /5/ uses a somewhat weaker definition of maximality according to which $f(\bar{x})$ need not be efficient.

2.3 Theorem 2.

Retaining the notation and assumptions of Theorem 1, let y^* be strictly positive over Y^+ and let x^o have the following properties: (1)$x^o \in X^c$, (2) $x \in X^c$ implies $y^*[f(x)] \leqq y^*[f(x^o)]$. Then x^o is maximal.

2.4 The conclusion of Theorem 1 would be easier to obtain if one only required that y^* be positive. However, if y^* is assumed positive (instead of strictly positive) in Theorem 2, maximality of x^o (in our sense of the term) does not follow. The weaker property which still does follow is what Slater calls maximality.

3. Scalar extremization: convexity versus differentiability.

In the remainder of this note we are concerned with the problem of maximizing the real-valued function $f(x)$ subject to the constraints $x \geqq 0$ and $g(x) \geqq 0$. The term "maximal" retains the meaning given to it in Theorem 1; since f is real-valued we are dealing with ordinary (scalar) maximization.

In sec. 4 we assume f and g to be continuous and concave but not necessarily differentiable. (The results can be extended to a broader class of linear topological spaces.) A Lagrangian saddle point is obtained.

In sec. 5 we assume f and g to be differentiable but not necessarily concave. The basic tool used is the concept of the Fréchet differential. (Whether a useful analogue exists outside Banach spaces I do not know.) The well-known necessary conditions for a Lagrangian saddle point are obtained. This result is of interest for problems involving "increasing returns."

4. Scalar maximization: convexity without differentiability.

4.1 Theorem 3. Let Y be the space of reals, X and Z Banach spaces; f and g concave continuous functions from X to Y and Z respectively; $X^+ \subseteq X$ a convex cone, $Z^+ \subseteq Z$ a closed convex cone with interior. Suppose further that, for some $x' \in X^+$, $g(x')$ is an interior point of Z^+.

The Lagrangian expression is given by

$$\emptyset(x, z^*) \equiv f(x) + z^*[\ g(x)\ ].$$

Let $\bar{x}$ be maximal. Then there exists a non-negative z_o^* such that

$$\emptyset(\bar{x}, z_o^*) \geqq \emptyset(x, z_o^*) \qquad \text{for all } x \geqq 0$$

and

$$\emptyset(\bar{x}, z_o^*) \leqq \emptyset(\bar{x}, z^*) \qquad \text{for all } z^* \geqq 0.$$

(When the last two relations hold we say that $\emptyset$ has a non-negative saddle point at $(\bar{x}, z_o^*)$.)

4.2 Theorem 4.

Let $\emptyset$ have a non-negative saddle point at (x^o, z_o^*) where z_o^* is non-negative; then x^o is maximal.

4.3 Combining Theorems 1, 3 on the one hand, and 2, 4 on the other we obtain the counterpart of Slater's Theorem 3 (s.p. equivalence).

5. Scalar extremization: differentiability without convexity.

5.0 Here we plagiarize Theorem 1 in Kuhn and Tucker, /4/. It will be recalled that the proof of /4/ is based on the Minkowski-Farkas Lemma, so that its counterpart for Banach spaces is needed here. [This matter was treated in CCDP 415 and 416; the latter paper corrected some of the deficiencies of 415, but it relied on a proof in "The Problem of Moments" by Shohat and Tamarkin in order to remove a certain restriction (cf. CCDP 416). Unfortunately, there is a gap in the proof of Shohat and Tamarkin and so far

I have not been able to verify it. The formulation of the Lemma given below is based on a somewhat different set of conditions and seems more satisfactory.]

5.1 Minkowski-Farkas Lemma.

Let X, Y be Banach spaces. $y' \geqq y''$ means $(y' - y'') \in Y^+$ where Y^+ is a closed convex cone in Y. T denotes a linear bounded transformation from X to Y, T* its adjoint. x*, y* are additive continuous functionals on X and Y respectively.

Define the following two sets in X^* (the conjugate space of X):

$$X_P^* \equiv \left\{ x^* : x^* = T^*(y^*),\ y^* \geqq 0 \right\} .$$

$$X_T^* \equiv \left\{ x^* : \text{for all } x,\ T(x) \geqq 0 \quad \text{implies } x^*(x) \geqq 0 \right\} .$$

Lemma. The two sets X_P^* and X_T^* are equal if and only if X_P^* is regularly convex. (The latter term is defined, p. 556, /3/.)

(The requirement of regular convexity of the set X_P^* is a restriction on the nature of T. In the finite-dimensional case this requirement is necessarily fulfilled.)

5.2 In what follows we shall use the concept of the Fréchet differential. (cf. /1/, pp. 71-3.) The differential of f(x) at $\bar{x}$ with increment x' is given by

$$d\, f(\bar{x}; x') \equiv \lim_{h \to 0} \frac{f(\bar{x} + hx') - f(\bar{x})}{h}$$

(For reasons of typography we use d instead of the customary δ.)

5.3 As in /4/, it is found essential to have a "constraint qualification" which is somewhat complex. This qualification is said to be satisfied at a point $\bar{x}$ if, for every non-zero increment (differential $x' \equiv x - \bar{x}$ satisfying the relations

$$x \geqq 0,$$

$$d\, g(\bar{x};\, x') + g(\bar{x}) \geqq 0,$$

there exists a differentiable function h from the closed interval 0,1 to X, say

$$x = h(t), \qquad 0 \leq t \leq 1,$$

such that

$$\bar{x} = h(0)$$

and

$$x' = d\, h(0;\, t') \quad \text{for some positive (real) } t'.$$

5.4 In applying the Minkowski-Farkas Lemma we find that to T in the Lemma corresponds $d\, g(\bar{x};\, x)$ with $\bar{x}$ fixed, so that x is the argument while the domain is in Z. We shall denote by G^* the set which is related to $d\, g(\bar{x};\, x)$ as X_P^* was to T. Of course, G^* must be regularly convex.

5.5 Theorem 5.

Let f, g be differentiable, G^* regularly convex (cf. 5.4) and let the "constraint qualification" of 5.3 be satisfied at the maximal point $\bar{x}$. Then the following relations are satisfied for some $z_0^* \geqq 0$:

(The subscripts following d indicate the variable of partial differentiation; ξ is an abbreviation for $(\bar{x},\, z_0^*)$ where all differentials are evaluated.)

$$d_x\, \phi(\xi\, ;\, x') \leqq 0 \qquad \text{for all } x \geqq 0 \quad (x \leq \bar{x} \leq x'\,),$$

$$d_x\, \phi(\xi\, ;\, \bar{x}\,) = 0,$$

$$d_{z^*}\phi(\xi\, ;\, z^{*'}) \equiv z^{*'}\,[g(\bar{x})] \leqq 0 \qquad \text{for all } z^* \geqq 0 \quad (z^* \leqq z_0^* \leqq z^{*'}),$$

$$d_{z^*}\, \phi(\xi\, ;\, z_0^*) = 0.$$

References

/1/ Hille: see Ref. /3/ in CCDP 415.

/2/ Krein and Rutman: see Ref. /4/ in CCDP 415.

/3/ Krein and Smulian, "On Regularly Convex Sets in the Space Conjugate to a Banach Space", Annals of Mathematics, Vol. 41, 1940, p. 556ff.

/4/ Kuhn and Tucker: see Ref. /5/ in CCDP 415.

/5/ Slater, M., "Lagrange Multipliers Revisited", CCDP 403.