Minkowski-Farkas Lemma in Banach Spaces

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October 17, 1952

1. In [2] a generalized version of the Minkowski-Farkas Lemma was stated, but the proof given was incorrect. Even assuming the set Z (defined in eq. (4) of [2]) to have been shown closed,[1] the additional assumption of reflexivity of the space X would have had to be made in order to deduce the result of [2] from equations (7) of [2].

2. The remainder of this paper is devoted to a (correct, one hopes) version of the Minkowski-Farkas Lemma which involves additional, though not very severe, conditions on the nature of the transformation T and/or the cone P_y. (We follow the notation and terminology of [2].) In Euclidean spaces the additional conditions are automatically satisfied if T is linear continuous and P_y the nonnegative orthant. Thus the present result contains the Minkowski-Farkas Lemma as given in [4].

3. Lemma. Let P_y be a convex cone in the Banach space Y and let T be a bounded linear transformation whose domain is the Banach space X and the range Y' is contained in Y. Assume furthermore that at least one of the following two conditions holds:

(a) $Y' = Y$;

(b) Y' is closed and P_y has interior.

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1. I am indebted to Professor B. Gelbaum, University of Minnesota, for pointing out the inadequacy of the argument in [2] concerning the closure of Z.

Then if x* is such that

(1) $x \in X, \quad T(x) \geq 0_y,$

imply

(2) $x^*(x) \geq 0,$

it follows that there exists a nonnegative functional y* satisfying the relation

(3) $x^* = T^*(y^*).$

4. <u>Proof.</u>

4.1. We first establish the fact that

(4) $T(x) = 0_y$ implies $x^*(x) = 0.$

Clearly, if

(5) $T(x) = 0_y,$

we have also

(6') $T(x) \geq 0_y$

and

(6") $T(-x) \geq 0_y.$

By hypothesis of the Lemma [(1) implies (2), applied successively to <u>x</u> and -x] we obtain from (6') and (6") respectively the inequalities

(7') $x^*(x) \geq 0,$

(7") $x^*(-x) \geq 0,$

hence

(8) $x^*(x) = 0.$

Thus (5) implies (8) which is the assertion of (4).

4.2. We now note that Y' is closed whether condition (a) or (b) of the Lemma is assumed to hold. Hence Banach's Théorème 8 ([1], p. 149) applies, and it follows that if x* satisfies (4) (which was shown in 4.1 to be the case), equation (3) possesses a solution y_0^*. However, we want a solution which is <u>nonnegative</u>. That such a solution exists is shown below.

4.3. We first observe that if y_0^* satisfies (3), we have

(9) $$y_0^*(y) \geqq 0 \quad \text{for } y \geqq 0_y,\ y \in Y'.$$

For, by definition of T^*, (3) is equivalent to

(10) $$x^*(x) = y^*[T(x)] \quad \text{for all } x \in X;$$

now suppose y_0^* satisfies (3), hence (10), while (9) is false. Then there exists an $x_0 \in X$ such that

(11) $$T(x_0) \geqq 0_y,$$

while

(12) $$y_0^*[T(x_0)] < 0.$$

In virtue of (10), we have

(13) $$x^*(x_0) = y_0^*[T(x_0)] < 0,$$

which contradicts the hypothesis of the Lemma, since (11) implies

(14) $$x^*(x_0) \geqq 0.$$

Thus (9) must hold.

4.4. Consider now the case (a), i.e., $Y' = Y$. It is clear that in this case (9) implies the nonnegativeness of y_0^*, since $y_0^*(y) \geqq 0$ for all $y \geqq 0_y$.

[It may be observed that the following holds:

When the conditions of the Lemma hold with (a) true (i.e., $Y' = Y$), the solution y^* of (3) is unique. (P_y need not have an interior.) This follows from Banach's Théorème 4, 1^o ([1], p. 148).]

4.5. It remains to consider the case $Y' \neq Y$ with condition (b) of the

Lemma holding.

We define here y^*_{oo} to be the linear bounded functional defined on Y' and such that

(15) $y^*_{oo}(y) = y^*_o(y)$ for all $y \in Y'$.

Noting that Y' is a linear subspace of Y, we shall show that y^*_{oo} can be extended to all of Y in such a way that it is nonnegative for all the nonnegative elements of Y. I.e., there exists a y^*_1 such that

(16) $y^*_1(y) = y^*_{oo}(y)$ for all $y \in Y'$

and

(17) $y^*_1(y) \geq 0$ for all $y \geq 0_y$, $y \in Y$.

4.6. Suppose first that Y' contains an interior point of P_y. Then the existence of y^*_1 is implied by Theorem 1.1 in [3] (p. 13). (The case $y^*_{oo}(y) = 0$ for all $y \in Y'$ is trivial, since y^*_1 equal to the null functional is a solution.)

But the condition that Y' contain an interior point of P_y can be dispensed with, as shown in [5] (p. XIII, 4.).

4.7. If the assumption that P_y has interior is abandoned, y^*_{oo} still has an extension satisfying equations (16) and (17); however, in this case we are only entitled to claim that y^*_1 is additive and homogeneous (as well as nonnegative), but not necessarily continuous.

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