1. In [5] Kuhn and Tucker use the Minkowski-Parkas Lemma (see Sec. 2 below) to prove the existence of a non-negative Lagrange multiplier vector for problems involving the extremization of a real-valued function of a finite-dimensional vector subject to a finite set of inequalities.

In the present note (Sec. 3 below) we prove a generalization of the Minkowski-Parkas Lemma which makes it possible to extend the Kuhn-Tucker results to a much broader class of problems where the unknown vector, as well as the constraints, are of a more general infinite-dimensional nature. This extension will be presented in a separate paper. It may only be mentioned here that in extending the Kuhn-Tucker results we apply our generalized Minkowski-Parkas Lemma to Lagrangian problems involving inequalities in a manner paralleling the procedure followed by Goldstine ([2]) who applies Theorem 4, p. 118 of Banach ([1]). (It should be noted, however, that our generalized Lemma is more closely related to Banach's Theorem

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8, p. 129, since, unlike Goldstine but like Kuhn and Tucker, we are not proving the uniqueness of the Lagrangian multiplier.)

2. The Minkowski-Farkas Lemma. (We state this Lemma in the formulation given by Kuhn and Tucker; the reader is also referred to [5] for bibliographical references to the original papers.)

An inequality \( b'x \geq 0 \) holds for all \( n \)-vectors \( x \) satisfying a system of \( m \) inequalities \( Ax \leq 0 \) only if \( b = A't \) for some \( m \)-vector \( t \geq 0 \).

(In the above statement \( A \) is an \( m \) by \( n \) matrix and a vector is said to be \( \geq 0 \) if and only if each of its components is nonnegative.)


3.1. Notation and terminology. (We largely follow the conventions of Hille [3] and Krein and Rutman [4]).

If \( W \) is a Banach space, we denote its elements by \( w \), the adjoint space (i.e., the space of linear bounded functionals \( w^* \) defined on \( W \)) by \( W^* \).

A set \( K \subseteq W \) is said to be a convex cone if \( w_1, w_2 \in K \) implies \( \lambda w_1 \in K \) for \( \lambda \geq 0 \) and \( w_1 + w_2 \in K \). The set \( \{ w^* : w^*(w) \geq 0 \text{ for all } w \in K \} \) in \( W^* \) is called the conjugate of \( K \) and is denoted by \( K^0 \). (It may be noted that \( K^0 \) is a closed convex cone, cf. [4], p. 12.)

We shall consider two Banach spaces \( X \) and \( Y \). \( P_y \) will denote a fixed arbitrarily chosen closed convex cone in \( Y \). We shall write \( y_1 \geq y_2 \) to mean \( y_1 - y_2 \in P_y \); in particular, \( y \geq 0_y \) (where \( 0_y \) is the null element of \( Y \)) means \( y \in P_y \). (It is easily seen that \( \geq \) is transitive and reflexive. \( P_y \) may be thought of as the nonnegative orthant of \( Y \).) The elements of the conjugate \( P^0_y \) of \( P_y \), where \( P^0_y = \{ y^* : y^*(y) \geq 0 \text{ for } y \geq 0_y \} \), will be said to be nonnegative; we shall write \( y^* \geq 0 \) to mean \( y^* \in P^0_y \).
T will denote a bounded linear transformation with the domain in X
and the range in Y. \( T^* \) is the adjoint of T, so that, by definition, if
\( x^* = T^*(y^*) \), we have \( x^*(x) = y^* \{ T(x) \} \) for all \( x \in X \).

3.2. Lemma. (The generalized Minkowski–Farkas Lemma).

If, for fixed T and \( x^* \),

(1) \( T(x) \notin O \) implies \( x^*(x) \geq 0 \) for all \( x \in X \),

then

(2) \( x^* = T^*(y^*) \) holds for some \( y^* \geq 0 \).

3.3. Proof of the Lemma.

We define the two sets

(3) \( X_T = \{ x : T(x) \neq O \} \),

and

(4) \( Z = \{ x^* : x^* = T^*(y^*) \text{ for some } y^* \geq 0 \} \),

and note that they are both closed convex cones.

[For if \( T(x_i) \in P_y \), \( i = 1, 2 \), then so are \( \lambda x_i \) for \( \lambda \leq 0 \)
and also \( T(x_1 + x_2) = T(x_1) + T(x_2) \), since T is linear bounded and \( P_y \) a convex cone; thus \( X_T \) is a convex cone. It is closed since a bounded linear transformation is continuous and hence \( x \in X_n \) implies \( T(x_n) \rightarrow T(x) \).

Similarly, if \( x^*_i \in Z \), \( i = 1, 2 \), so are \( \lambda x^*_i \) for \( \lambda \leq 0 \) and \( x^*_1 + x^*_2 \), since \( y^*_i \geq 0 \), \( i = 1, 2 \), implies \( \lambda y^*_i = 0 \) for \( \lambda \leq 0 \) and \( y^*_1 + y^*_2 \geq 0 \), so that \( Z \) is a convex cone; it is closed because the adjoint \( T^* \) of a linear bounded transformation T is continuous and hence an argument analogous to that for \( X_T \) can again be used.]

The lemma being proved may now be restated as follows: if \( x^* \in X_T^\theta \),
then \( x^* \in Z \), i.e., \( x_T^\theta \in Z \). We shall show that, in fact,

(5) \( x_T^\theta = Z \).

We use Corollary 1.3, p. 16, in [1] which states that \( w_0 \in W \) is an
element of the closure $\bar{K}$ of the convex cone $K$ if and only if

$$w^*(w_o) \preceq 0 \text{ for all } w^* \in K^\circ.$$ 

When $K$ is closed this becomes

$$(6)\quad w_o \in K \text{ if and only if } w^*(w_o) \preceq 0 \text{ for all } w^* \in K^\circ.$$ 

We shall use (6) to prove (5). We shall show that

$$(7')\quad x_o \in X_T \text{ implies } x^*(x_o) \preceq 0 \text{ for all } x^* \in Z,$$

and

$$(7'')\quad x_o \notin X_T \text{ implies the existence of an } x^* \in Z \text{ such that } x^*(x_o) \prec 0.$$ 

Since $X_T$ is closed, relations (7) yield (5) (in view of (6)) and the proof will be complete when (7) has been proved.

Using the definitions of $T^*$ and $Z$, we have

$$x^*(x) = y^* [T(x)] \quad y^* \preceq 0$$

for all $x \in X$.

Now let $x_o \in X_T$. Then $T(x_o) \preceq 0_y$, hence $y^* [T(x_o)] \preceq 0$, so that (7') follows. On the other hand suppose $x_o \notin X_T$. Then $y_o = T(x_o) \not\in 0_y$, and this implies the existence of a $y^*$ effective in (7''). To see this we recall that the elements $y^* \preceq 0$ form the cone $P_y^\circ$ conjugate to the closed convex cone $P_y$; hence, again in virtue of Corollary 1.3, p. 16 in [4], $y^*(y_o) \preceq 0$ for all $y^* \in P_y^\circ$ only if $y_o \in P_y$.

3.4. The H"{o}nffki-S"{o}rkas Lemma is a special case of the Lemma in 3.2 with the spaces $X$ and $Y$ respectively $n$- and $m$-dimensional and $P_y$ defined as the nonnegative orthant in the space $Y$. In the statement of Sec. 2 above $b$ is an $x^*$, $t$ a (nonnegative) $y^*$. The transpose $A'$ of $A$ corresponds to the adjoint $T^*$ of $T$. 
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