THE MINKOWSKI-FARKAS LEMMA FOR BOUNDED LINEAR TRANSFORMATIONS IN BANACH SPACES

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1. In [5] Kuhn and Tucker use the Minkowski-Farkas Lemma (see Sec. 2 below) to prove the existence of a nonnegative Lagrange multiplier vector for problems involving the extremization of a real-valued function of a finite-dimensional vector subject to a finite set of inequalities.

In the present note (Sec. 3 below) we prove a generalization of the Minkowski-Farkas Lemma which makes it possible to extend the Kuhn-Tucker results to a much broader class of problems where the unknown vector, as well as the constraints, are of a more general infinite-dimensional nature. This extension will be presented in a separate paper. It may only be mentioned here that in extending the Kuhn-Tucker results we apply our generalized Minkowski-Farkas Lemma to Lagrangian problems involving inequalities in a manner paralleling the procedure followed by Goldstine ([2]) who applies Theorem 4, p. 148 of Banach ([1]). (It should be noted, however, that our generalized Lemma is more closely related to Banach's Theorem

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8, p. 149, since, unlike Goldstine but like Kuhn and Tucker, we are not proving the uniqueness of the Lagrangian multiplier.)

2. The Minkowski-Farkas Lemma. (We state this Lemma in the formulation given by Kuhn and Tucker; the reader is also referred to [5] for bibliographical references to the original papers.)

An inequality $b'x \geqq 0$ holds for all n-vectors x satisfying a system of m inequalities $Ax \geqq 0$ only if $b = A't$ for some m-vector $t \geqq 0$.

(In the above statement A is an m by n matrix and a vector is said to be $\geqq 0$ if and only if each of its components is nonnegative.)

3. The Generalized Minkowski-Farkas Lemma.

3.1. Notation and terminology. (We largely follow the conventions of Hille [3] and Krein and Rutman [4]).

If W is a Banach space, we denote its elements by w, the adjoint space (i.e., the space of linear bounded functionals w^* defined on W) by W^*.

A set $K \subseteq W$ is said to be a convex cone if $w_1, w_2 \in K$ implies $\lambda w_1 \in K$ for $\lambda \geqq 0$ and $w_1 + w_2 \in K$. The set $\{w^* : w^*(w) \geqq 0 \text{ for all } w \in K\}$ in W^* is called the conjugate of K and is denoted by $K^{\oplus}$. (It may be noted that $K^{\oplus}$ is a closed convex cone, cf. [4], p. 12.)

We shall consider two Banach spaces X and Y. P_y will denote a fixed arbitrarily chosen closed convex cone in Y. We shall write $y_1 \geqq y_2$ to mean $y_1 - y_2 \in P_y$; in particular, $y \geqq 0_y$ (where 0_y is the null element of Y) means $y \in P_y$. (It is easily seen that $\geqq$ is transitive and reflexive. P_y may be thought of as the nonnegative orthant of Y.) The elements of the conjugate $P_y^{\oplus}$ of P_y, where $P_y^{\oplus} = \{y^* : y^*(y) \geqq 0 \text{ for } y \geqq 0_y\}$, will be said to be nonnegative; we shall write $y^* \geqq 0$ to mean $y^* \in P_y^{\oplus}$.

T will denote a bounded linear transformation with the domain in X and the range in Y. T^* is the adjoint of T, so that, by definition, if $x^* = T^*(y^*)$, we have $x^*(x) = y^* [T(x)]$ for all $x \in X$.

3.2. Lemma. (The generalized Minkowski-Farkas Lemma).

If, for fixed T and x^*,

$$(1) \qquad T(x) \geqq 0_y \text{ implies } x^*(x) \geqq 0 \text{ for all } x \in X,$$

then

$$(2) \qquad x^* = T^*(y^*) \text{ holds for some } y^* \geqq 0.$$

3.3. Proof of the Lemma.

We define the two sets

$$(3) \qquad X_T \equiv \{x : T(x) \geqq 0_y\}$$

and

$$(4) \qquad Z \equiv \{x^* : x^* = T^*(y^*) \text{ for some } y^* \geqq 0\}$$

and note that they are both closed convex cones.

[For if $T(x_i) \in P_y$, $i = 1, 2$, then so are $T(\lambda x_i) = \lambda T(x_i)$ for $\lambda \geqq 0$ and also $T(x_1+x_2) = T(x_1) + T(x_2)$, since T is linear bounded and P_y a convex cone; thus X_T is a convex cone. It is closed since a bounded linear transformation is continuous and hence $x_n \rightarrow x$ implies $T(x_n) \rightarrow T(x)$.

Similarly, if $x_i^* \in Z$, $i = 1, 2$, so are λx^* $(\lambda \geqq 0)$ and $x_1^* + x_2^*$, since $y_i^* \geqq 0$, $i = 1, 2$, implies $\lambda y_i^* \geqq 0$ $(\lambda \geqq 0)$ and $y_1^* + y_2^* \geqq 0$, so that Z is a convex cone; it is closed because the adjoint T^* of a linear bounded transformation T is continuous and hence an argument analogous to that for X_T can again be used.]

The lemma being proved may now be restated as follows: if $x^* \in X_T^\theta$, then $x^* \in Z$, i.e., $X_T^\theta \subseteq Z$. We shall show that, in fact,

$$(5) \qquad X_T^\theta = Z.$$

We use Corollary 1.3, p. 16, in [4] which states that $w_0 \in W$ is an element of the closure $\bar{K}$ of the convex cone K if and only if

$$w^*(w_o) \geqq 0 \quad \text{for all } w^* \in K^{\oplus} .$$

When K is closed this becomes

(6) $$w_o \in K \text{ if and only if } w^*(w_o) \geqq 0 \text{ for all } w^* \in K^{\oplus} .$$

We shall use (6) to prove (5). We shall show that

(7') $$x_o \in X_T \text{ implies } x^*(x_o) \geqq 0 \quad \text{for all } x^* \in Z,$$

and

(7") $$x_o \notin X_T \text{ implies the existence of an } x^* \in Z \text{ such that } x^*(x_o) < 0.$$

Since X_T is closed, relations (7) yield (5) (in view of (6)) and the proof will be complete when (7) has been proved.

Using the definitions of T^* and Z, we have

$$x^*(x) = y^* [T(x)] \qquad y^* \geqq 0$$

for all $x \in X$.

Now let $x_o \in X_T$. Then $T(x_o) \leqq 0_y$, hence $y^* [T(x_o)] \leqq 0$, so that (7') follows. On the other hand suppose $x_o \notin X_T$. Then $y_o \equiv T(x_o) \nleqq 0_y$, and this implies the existence of a y^* effective in (7"). To see this we recall that the elements $y^* \geqq 0$ form the cone $P_y^{\oplus}$ conjugate to the closed convex cone P_y; hence, again in virtue of Corollary 1.3, p. 16 in [4], $y^*(y_o) \leqq 0$ for all $y^* \in P_y^{\oplus}$ only if $y_o \in P_y$.

3.4. The Minkowski-Farkas Lemma is a special case of the Lemma in 3.2 with the spaces X and Y respectively n- and m-dimensional and P_y defined as the nonnegative orthant in the space Y. In the statement of Sec. 2 above b is an x^*, t a (nonnegative) y^*. The transpose A' of A corresponds to the adjoint T^* of T.

REFERENCES

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