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Definite and Semi-Definite Quadratic Forms[1/] *

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Conditions for a quadratic form to be definite or semi-definite, with or without linear constraints, are very frequently used in classical economic theories; however it is difficult, when not impossible, to find short and complete proofs in the literature. Original proofs are given here in a unified treatment of the subject.

x, A, B are matrices of orders $n.1$, $n.n$, $n.m$. M being a matrix, $M_{p.q}$ is obtained from M by keeping only the elements in the first p rows and the first q columns; M_p stands for $M_{p.p}$; when M is square, $|M|$ is its determinant. Primed letters denote transposes.

1. Definite Quadratic Forms

m_{ij} is the i^{th} row, j^{th} column element of M; x_i stands for x_{i1}; L_r is a linear form of the variables x_i, $r \leq i \leq n$, whose coefficient of x_r is unity. As a matter of convention, $|A_o| = 1$.

Th. 1. Let A be symmetric. $x'Ax = \sum_{r=1}^{n} \frac{|A_r|}{|A_{r-1}|}(L_r)^2$ if and only if $|A_r| \neq 0$ for $r = 1, \ldots, n-1$.

Necessity: Obvious

Sufficiency: Perform on the quadratic form $Q_1(x_1, \ldots, x_n) = x'Ax$ a

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standard decomposition into squares and assume that after the first $r - 1$ steps Q_1 has been written under the form

$$(1) \quad Q_1(x_1, \ldots, x_n) = \sum_{i,j} a_{ij}\, x_i\, x_j = c_1(L_1)^2 + \ldots + c_{r-1}(L_{r-1})^2 + Q_r(x_r, \ldots, x_n)$$

where $c_i \neq 0$ for $i = 1, \ldots, r - 1$, and Q_r is a quadratic form of the variables x_i, $r \leqq i \leqq n$. We want to find the coefficient c_r of x_r^2 in Q_r.

Set $x_i = 0$ for $i = r + 1, \ldots, n$, and then derive from (1) the r identities

$$\frac{1}{2}\, \frac{\partial Q_1}{\partial x_i} = \sum_{j=1}^{r} a_{ij}\, x_j = \sum_{j=1}^{i} c_j\, L_j\, \frac{\partial L_j}{\partial x_i} \qquad \text{for } i = 1, \ldots, r - 1$$

$$\frac{1}{2}\, \frac{\partial Q_1}{\partial x_r} = \sum_{j=1}^{r} a_{rj}\, x_j = \sum_{j=1}^{r-1} c_j\, L_j\, \frac{\partial L_j}{\partial x_r} + c_r\, x_r .$$

Make $L_j = 0$ for $j = 1, \ldots, r - 1$, this system has a nonzero solution and therefore also

$$\sum_{j=1}^{r} a_{ij}\, x_j = 0 \qquad i = 1, \ldots, r - 1$$

$$\sum_{j=1}^{r} a_{rj}\, x_j - c_r\, x_r = 0$$

i.e., $\begin{vmatrix} A_{r-1} & a_{ir} \\ a_{rj} & a_{rr} - c_r \end{vmatrix} = 0$ or $c_r = \dfrac{|A_r|}{|A_{r-1}|}$.

Since $|A_r| \neq 0$, it is possible to perform the r^{th} step.

<u>Th. 2. Let A be symmetric.</u> $x'Ax > 0$ (<u>resp.</u> < 0) <u>for every</u> $x \neq 0$ <u>if and only if</u> $\|A_r\| > 0$ (<u>resp.</u> $(-1)^r \|A_r\| > 0$) <u>for</u> $r = 1, \ldots, n$.

If $Ax = 0$ had a nonzero solution x_0, one would have $x_0' Ax_0 = 0$, and the quadratic form would not be definite. It is therefore necessary that $|A| \neq 0$ and more generally $\|A_r\| \neq 0$ for $r = 1, \ldots, n$ (set $x_{r+1}, \ldots, x_n$ all equal to zero; the quadratic form $Q_1(x_1, \ldots, x_r, 0, \ldots, 0)$ must also be definite).

A straightforward application of Th. 1. then proves the statement.

2. Quadratic Forms Definite Under Linear Constraints

Th. 3. $x'Ax \;\; 0$ (resp. < 0) for every $x \neq 0$ such that $B'x = 0$ if and only if there exists a number λ such that $x'Ax + \lambda x'BB'x$ is a positive (resp. negative) definite quadratic form.

Sufficiency: Obvious

Necessity: The function $y(x) = -\dfrac{x'Ax}{x'BB'x}$ is continuous on the set $\{x: x'x = 1 \text{ and } B'x \neq 0\}$, and tends to $-\infty$ (resp. $+\infty$) whenever x tends to a boundary point; it has therefore a finite maximum (resp. minimum) λ^*. Any $\lambda \geq \lambda^*$ (resp. $< \lambda^*$) has the desired property.

Lemma $|A + \lambda BB'|$ is a polynomial in λ whose term of highest order (possibly null) is $(-1)^m \begin{vmatrix} A, & B \\ B' & O_m \end{vmatrix} \lambda^m$.

From $\begin{bmatrix} A & \lambda B \\ B' & -I_m \end{bmatrix} \begin{bmatrix} I_n & O_{nm} \\ B' & I_m \end{bmatrix} = \begin{bmatrix} A + \lambda BB' & \lambda B \\ O_{m.n} & -I_m \end{bmatrix}$ follows $\begin{vmatrix} A & \lambda B \\ B' & -I_m \end{vmatrix}$ $= (-1)^m |A + \lambda BB'|$. In the development of the left-hand determinant a term contains the highest possible power of λ if in every one of the last m columns one takes an element of λB. Such terms are unaffected if $-I_m$ is replaced by any other m.m matrix: take O_m.

Th. 4. Let A be symmetric and $|B_m|$ be different from zero. $x'Ax > 0$ for every $x \neq 0$ such that $B'x = 0$ if and only if $(-1)^m \begin{vmatrix} A_r & B_{rm} \\ B'_{rm} & 0 \end{vmatrix} > 0$ for $r = m + 1, \ldots, n$.

Necessity: Consider the equations $\begin{cases} Ax + B\xi = 0 \\ B'x = 0 \end{cases}$ where ξ is a m.1 matrix. A solution $\begin{bmatrix} x \\ \xi \end{bmatrix}$ is such that $x'Ax + x'B\xi = 0$, i.e., $x'Ax = 0$. This must imply $x = 0$, therefore $B\xi = 0$, and, since $|B_m| \neq 0$, $\xi = 0$. The system must have no other solution than 0, i.e., $\begin{vmatrix} A & B \\ B' & 0 \end{vmatrix} \neq 0$.

From Th. 3. and Th. 2. for every $\lambda > \lambda^*$ one must have $|A + \lambda BB'| \geq 0$.

From the lemma one must have $(-1)^m \begin{vmatrix} A & B \\ B' & 0 \end{vmatrix} > 0$.

This argument can be made for any r, $m \leq r \leq n$.

<u>Sufficiency</u>: I shall prove that the coefficient of the term of highest order in λ of $|A_r + \lambda B_{rm} B'_{rm}|$ is positive whatever be $r \leq n$. It will therefore be possible to choose λ large enough to make those n leading minors positive and consequently $[A + \lambda BB']$ positive definite (Th. 2.)

a) If $r > m$, it is true by assumption.

b) If $r \leq m$, the development technique used in the proof of the lemma shows that every term of $\begin{vmatrix} A_r & \lambda B_{rm} \\ B'_{rm} & -I_m \end{vmatrix}$ of order higher than r vanishes. The r^{th} order term is $\Sigma (-1)^{m-r} \begin{vmatrix} A_r & \tilde{B}_r \\ \tilde{B}'_r & 0 \end{vmatrix} \lambda^r$ where $\tilde{B}_r$ is any r.r sub-matrix of B_{rm} whose columns are in the natural order. This term is equal to $(-1)^m \lambda^r \Sigma |\tilde{B}_r|^2$, and finally the coefficient of λ^r in $|A_r + \lambda B_{rm} B'_{rm}|$ is $\Sigma |\tilde{B}_r|^2$ which cannot vanish since $|B_m| \neq 0$.

A similar argument proves

<u>Th. 5. Let A be symmetric and $|B_m|$ be different from zero.</u> $x'Ax < 0$ <u>for every</u> $x \neq 0$ <u>such that</u> $B'x = 0$ <u>if and only if</u> $(-1)^r \begin{vmatrix} A_r & B_{rm} \\ B'_{rm} & 0 \end{vmatrix} > 0$ <u>for</u> $r = m + 1, \ldots, n$.

<u>3. Semi-Definite Quadratic Forms</u>

Denote by π a permutation of the first n integers, A^{π} the matrix obtained from A by performing the permutation π on its rows and on its columns, B^{π} the matrix obtained from B by performing the permutation π on its rows.

<u>Th. 6.</u> $x'Ax \geq 0$ (<u>resp.</u> ≤ 0) <u>for every</u> x <u>if and only if</u> $x'Ax + \alpha x'x > 0$ (<u>resp.</u> < 0) <u>for every</u> $x \neq 0$ <u>and every</u> $\alpha > 0$ (<u>resp.</u> < 0).

<u>Necessity</u>: Obvious

<u>Sufficiency</u>: Obvious by a continuity argument.

Th. 7. Let A be symmetric. $x'Ax \gtreqless 0$ (resp. $\leqq 0$) for every x if and only if $|A_r^{\pi}| \geqq 0$ (resp. $(-1)^r |A_r^{\pi}| \geqq 0$) whatever be $r = 1, \ldots, n$ and π.

Necessity: From Th. 6. and Th. 2 $|A_r^{\pi} + \alpha I_r| > 0$ whatever be $r = 1, \ldots, n, \alpha > 0$, and π. This implies $|A_r^{\pi}| \geqq 0$ whatever be $r = 1, \ldots, n$ and π.

Sufficiency: $|A_r + \alpha I_r| = \alpha^r + \sum_{i=0}^{r-1} \alpha^i S_{r-i}^r$ where S_i^r is the sum of all the principal minors of A_r of order i. From the assumption every S_i^r is nonnegative and, if α is positive, the right-hand member is positive. This implies (Th. 2) that the quadratic form $x'Ax + \alpha x'x$ is positive definite whatever be $\alpha > 0$. An application of Th. 6. yields the result.

The result for negative forms is proved by a transposed argument.

4. Quadratic Forms Semi-Definite Under Linear Constraints

The same techniques yield

Th. 8. $x'Ax \geqq 0$ (resp. $\leqq 0$) for every x such that $B'x = 0$ if and only if $x'Ax + \alpha x'x > 0$ (resp. < 0) for every $\alpha > 0$ (resp. < 0) and every $x \neq 0$ such that $B'x = 0$.

Th. 9. Let A be symmetric and $|B_m|$ be different from zero. $x'Ax \geqq 0$ for every x such that $B'x = 0$ if and only if $(-1)^m \begin{vmatrix} A_r^{\pi} & B_{rm}^{\pi} \\ B_{rm}^{\pi\prime} & 0 \end{vmatrix} \geqq 0$ whatever be $r = m + 1, \ldots, n$ and π.

(Note that now the term of highest order in α in the development of $(-1)^m \begin{vmatrix} A_r + \alpha I_r & B_{rm} \\ B_{rm}' & 0 \end{vmatrix}$ is $\alpha^{r-m} \Sigma |\tilde{B}_m|^2$ where $\tilde{B}_m$ is any m.m submatrix of B_{rm} whose rows are in the natural order.)

Th. 10. Let A be symmetric and $|B_m|$ be different from zero. $x'Ax \leqq 0$ for every x such that $B'x = 0$ if and only if $(-1)^r \begin{vmatrix} A_r^{\pi} & B_{rm}^{\pi} \\ B_{rm}^{\pi\prime} & 0 \end{vmatrix} \geqq 0$ whatever be $r = m + 1, \ldots,$ and π.

5. Historical Note[2]

The idea of the decomposition into squares of Th. 1. goes back to Lagrange [4] who however performed it only in the cases n = 2, 3 and did not conjecture the general form of the coefficient $\frac{|A_r|}{|A_{r-1}|}$. This decomposition was given for the first time in all its generality by F. Brioschi [2].

M. Allais [1] gave a proof for a particular case of Th. 5., the one where m = 1 and all the elements of B are equal to unity. The most complete (but unduly long) proof for Th. 4. and 5 was given by J. Seitz [7].

The method used by P. Samuelson [6, p. 376-378] and which goes back to Richelot [5], consists in studying the roots of the equation in λ, $\begin{vmatrix} A - \lambda I_n & B \\ B' & 0 \end{vmatrix} = 0$. It proves easily Th. 9-10 and the <u>necessity</u> of the conditions given in Th. 4-5 but it does not prove that it is sufficient that the <u>leading</u> minors of $\begin{vmatrix} A, & B \\ B' & 0 \end{vmatrix}$ be of the proper signs.

Similarly the study [6, p. 370-375] of the roots of $|A - \lambda I| = 0$ proves easily Th. 7. and the necessity of the conditions of Th. 2.; it cannot prove without additional steps (see f. ex. [3]) their sufficiency.

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2. All statements below are made "to the best of my knowledge." I would be very grateful for significant references which I might have overlooked.

References

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