Quadratic Forms Definite Under Linear Constraints[1]

Gerard Debreu

August 13, 1951

This paper offers a proof of a theorem whose applications in the classical theories of economic equilibrium are numerous.

x, A, B are matrices of orders n.1, n.n, n.m. M being a matrix, $M_{p \cdot q}$ is obtained from M by keeping only the elements in the first p rows and the first q columns; M_p stands for $M_{p \cdot p}$. Primed letters denote transposes.

Th. 1 <u>$x'Ax > 0$ for every $x \neq 0$ such that $B'x = 0$ if and only if there exists a number λ such that $x'Ax + \lambda x'BB'x$ is a positive definite quadratic form.</u>

It is sufficient.

Now, the function $y(x) = -\dfrac{x'Ax}{x'BB'x}$ is continuous on the set $\{x: x'x = 1 \text{ and } B'x \neq 0\}$, and tends to $-\infty$ whenever x tends to a boundary point; it has therefore a finite maximum λ^*. Any $\lambda > \lambda^*$ has the desired property.

Th. 2 <u>$|A + \lambda BB'|$ is a polynomial in λ whose term of highest order (possibly null) is</u> $(-1)^m \begin{vmatrix} A & B \\ B' & O_{m.m} \end{vmatrix} \lambda^m$.

---

1. My acknowledgments are due to J. Marschak whose interest in this problem made me initiate this paper and to M. Slater with whom I had most valuable discussions.

From $\begin{bmatrix} A & \lambda B \\ B' & -I_m \end{bmatrix} \begin{bmatrix} I_n & 0_{n\cdot m} \\ B' & I_m \end{bmatrix} = \begin{bmatrix} A+\lambda BB' & \lambda B \\ 0_{m\cdot n} & -I_m \end{bmatrix}$ follows $\begin{vmatrix} A & \lambda B \\ B' & -I_m \end{vmatrix} = (-1)^m \left| A+\lambda BB' \right|$

In the development of the left-hand determinant a term contains the highest possible power of λ if in every one of the last m columns one takes an element of λB. Such terms are unaffected if $-I_m$ is replaced by any other $m\cdot m$ matrix: take $0_{m\cdot m}$.

Th. 3 <u>Let A be symmetric and $|B_{mm}|$ be different from zero. $x'Ax > 0$ for every $x \neq 0$ such that $B'x = 0$ if and only if</u>

$$(-1)^m \begin{vmatrix} A_r & B_{rm} \\ B'_{rm} & 0 \end{vmatrix} > 0 \qquad \underline{\text{for } r = m+1,\ldots,n.}$$

1) Necessity: Consider the equations $\begin{cases} Ax + B\xi = 0 \\ B'x = 0 \end{cases}$ where ξ is an $m\cdot 1$ matrix. A solution $\begin{bmatrix} x \\ \xi \end{bmatrix}$ is such that $x'Ax + x'B\xi = 0$, i.e., $x'Ax = 0$. This must imply $x = 0$, therefore $B\xi = 0$, and, since $|B_{m\cdot m}| \neq 0$, $\xi = 0$. The system must have no other solution than 0, i.e., $\begin{vmatrix} A & B \\ B' & 0 \end{vmatrix} \neq 0$.

From th. 1, for every $\lambda > \lambda^*$ one must have $|A+\lambda BB'| > 0$.

From th. 2 one must have $(-1)^m \begin{vmatrix} A & B \\ B' & 0 \end{vmatrix} > 0$.

This argument can be made for any r, $m \leqq r \leqq n$.

2) Sufficiency: I shall prove that the coefficient of the term of highest order in λ of $|A_r + \lambda B_{rm} B'_{rm}|$ is positive whatever be $r \leqq n$.
It will therefore be possible to choose λ large enough to make these n leading minors positive and consequently $[A+\lambda BB']$ positive definite.

a) If $r > m$, it is true by assumption.

b) If $r \leqq m$, the development technique used in the proof of th. 2 shows that every term of $\begin{vmatrix} A_r & \lambda B_{rm} \\ B'_{rm} & -I_m \end{vmatrix}$ of order higher than r vanishes.

The r^{th} order term is $\Sigma(-1)^{m-r} \begin{vmatrix} A_r & \tilde{B}_r \\ \tilde{B}'_r & 0 \end{vmatrix} \lambda^r$ where $\tilde{B}_r$ is any $r \cdot r$ submatrix of $B_{r \cdot m}$ whose columns are in the natural order.

This term is equal to $(-1)^m \lambda^r \Sigma \left|\tilde{B}'_r\right|^2$, and finally the coefficient of λ^r in $\left|A_r + \lambda B_{rm} B'_{rm}\right|$ is $\Sigma\left|\tilde{B}_r\right|^2$ which cannot vanish since $\left|B_{mm}\right| \neq 0$.

A similar argument proves

Th. 4 <u>Let A be symmetric and $\left|B_{mm}\right|$ be different from zero. $x'Ax < 0$ for every $x \neq 0$ such that $B'x = 0$ if and only if</u> $(-1)^r \begin{vmatrix} A_r & B_{rm} \\ B'_{rm} & 0 \end{vmatrix} > 0$ <u>for $r = m+1, \ldots, n$.</u>