A THEOREM ON CHARACTERISTIC ROOTS

by Morton Slater

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The purpose of this note is to give a simple proof of Theorem 3 of Economics 2016 "The Stability of Systems with Non-negative Coefficients," by John Chipman.

We use the following notation: "e" will denote a column vector (of appropriate length) all of whose components are equal to one.

Theorem Given $M \geqq 0$, $e'M \leqq e'$

Assertion: The characteristic roots of M have modulus $\leqq 1$, and either all roots have modulus < 1 or 1 is a root.

Proof: We use two facts, the first of which was also used by Chipman:

1^o If the max of the column sums of M is $\leqq 1$, then all the characteristic roots of M are of modulus ≤ 1. If the max is < 1, then all the characteristic roots have modulus < 1. (See Waugh)

2^o If λ is a characteristic root of M, then λ^k is a characteristic root of M^k, where k is a positive integer.

(This is completely elementary).

To proceed with the proof:

By hypothesis $e' M \leqq e'$. By induction, since $M \geqq 0$,

$e' M^{n+1} \leqq e' M^n$ (coordinatewise) , for all positive integral n.
Since $e' M^n$ is $\geqq 0$ for all n, we see that

$$e' M^n \text{ decreases to } q' \text{ (say)} \geqq 0.$$

If $q' = 0$, then for some n $e' M^n < e'$ and so by 1° the characteristic roots of M^n are of modulus <1. Hence by 2° the characteristic roots of M are also <1 in absolute value.

If $q' \geq 0$, then since

$$\lim_{n \to \infty} e' M^n M = \lim_{n \to \infty} e' M^{n+1} = \lim_{n \to \infty} e' M^n$$

we have $$q' M = q'$$

whence M' has a characteristic root 1 and hence M also has a characteristic root 1. q.e.d.

Reference:

F. V. Waugh "Inversion of the Leontief Matrix by Power Series," Econometrica Volume 8 No. 2.