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A THEOREM ON CHARACTERISTIC ROOTS

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The purpose of this note is to give a simple proof of Theorem 3 of Economics 2016 "The Stability of Systems with Non-negative Coefficients," by John Chipman.

We use the following notation: "e" will denote a column vector (of appropriate length) all of whose components are equal to one.

Theorem Given $M \geq 0$, $e'M \leq e'$

Assertion: The characteristic roots of M have modulus ≤ 1 , and either all roots have modulus < 1 or 1 is a root.

Proof: We use two facts, the first of which was also used by Chipman:

- 1° If the max of the column sums of M is ≤ 1 , then all the characteristic roots of M are of modulus ≤ 1 . If the max is < 1 , then all the characteristic roots have modulus < 1 . (See Waugh)
- 2° If λ is a characteristic root of M, then λ^k is a characteristic root ^{of} M^k , where k is a positive integer.

(This is completely elementary).

To proceed with the proof:

By hypothesis $e'M \leq e'$. By induction, since $M \geq 0$,

$e' M^{n+1} \leq e' M^n$ (coordinatewise), for all positive integral n .

Since $e' M^n$ is ≥ 0 for all n , we see that

$e' M^n$ decreases to q' (say) ≥ 0 .

If $q' = 0$, then for some n $e' M^n < e'$ and so by 1^o the characteristic roots of M^n are of modulus < 1 . Hence by 2^o the characteristic roots of M are also < 1 in absolute value.

If $q' > 0$, then since

$$\lim_{n \rightarrow \infty} e' M^n M = \lim_{n \rightarrow \infty} e' M^{n+1} = \lim_{n \rightarrow \infty} e' M^n$$

we have

$$q' M = q'$$

whence M' has a characteristic root 1 and hence M also has a characteristic root 1. q.e.d.

Reference:

F. V. Waugh "Inversion of the Leontief Matrix by Power Series," Econometrica Volume 8 No. 2.