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I. Introduction

1.1. Two technological circumstances go a long way toward explaining the economy of holding inventories: the presence of a fixed cost in ordering or manufacturing a quantity of a commodity; and the lags between orders and receipts, or the beginning and the completion of a manufacturing process, as the case may be. Of these two factors the first has received most attention in formal models of optimal inventory policy. To our knowledge the literature does not contain any treatment, at once rigorous and practical, of optimal inventory policy under uncertainty for the case of substantial delivery lags. Assume for the moment that decisions can be taken only at the beginning of each time interval of unit length (normally a day). The equation system proposed by Dvoretzsky, Kiefer and Wolfowitz [2, pp. 211-213] for the case of fixed lags of T time units are of a high order of complexity, except in the

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particular case T = 1. If T is very large, say of the order of 100 or even 10 time units, this method ceases to be useful. A different approach appears to be called for. This paper explores the inventory problem with long lags for the simplest case involving uncertainty of demand.

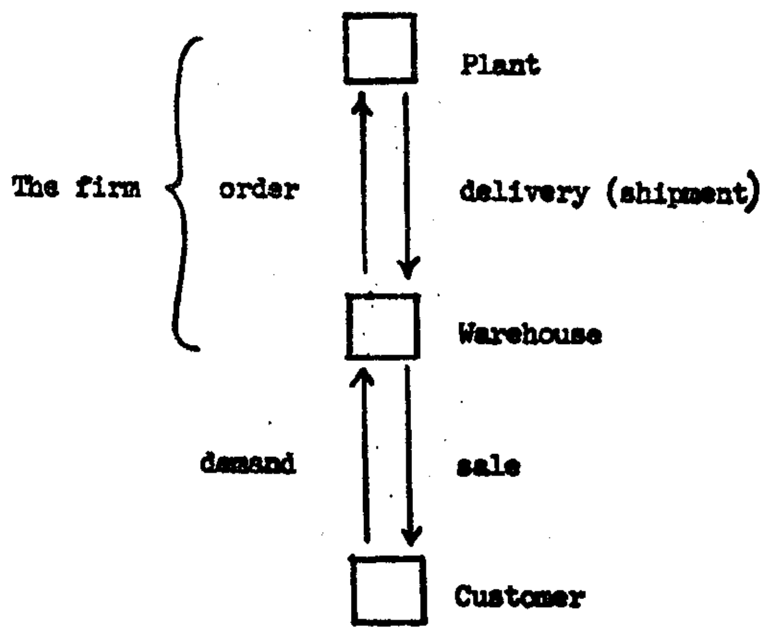

Fig. 1

1.2. This case is characterized as follows. (Fig. 1)

1.2.1. Demand (of the customer)

1.2.1.1. Demand is described by a Poisson process. This means: the probability of more than one unit being demanded at a time is zero; for a small period the probability of a demand for one unit is proportional to the length of time with a proportionality factor independent of time. It is well known [3, pp. 364-367] that under these conditions the time intervals between successive demands obey a negative-exponential distribution and that the number of units demanded during any time interval of fixed length is subject to a Poisson distribution.

1.2.1.2. Demand not satisfied immediately because of a shortage of stock is postponed.* Therefore a backlog of demand may be run up. No limitation will be imposed on the admissible backlog.

1.2.2. <u>Ordering</u> (by the warehouse manager)

1.2.2.1. Ordering may take place at any time, i.e. not only at the beginning of a period as assumed in the previous papers [1] [2] on inventory policy. For convenience of language we shall use the time unit "day". There is no implication, however, that transactions should occur only at the beginning of the day.

1.2.2.2. Delivery follows upon orders by a certain lag, defined for ease of expression in terms of days. This lag need not be an integer however.

1.2.3. <u>Costs</u>

These consist of three parts: the cost of ordering, the carrying cost and the cost of deferred demand.

1.2.3.1. The cost of ordering is assumed to be linear: it consists of a fixed part which is independent of the order size, and of a variable part which is proportional to the amount ordered. All ordering costs are charged to the firm at the time at which the shipment is received, i.e. at which it materializes into a stock.

1.2.3.2. The carrying cost of the inventory is assumed to be proportional to the maximal amount of inventory on hand and ordered. This assumption is most appropriate when the warehouse and the plant are controlled by the same enterprise.

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* From the further description of the problem it will become clear that no customer has to wait longer than the delivery lag while in general his waiting time will be a fraction of this.

1.2.3.3. The cost of deferred demand -- consisting of losses in goodwill or concessions made to waiting customers -- is assumed to consist of a fixed part per day of shortage and a variable part per day proportional to the amount of the backlog.

1.2.4. The aim of inventory policy as assumed here is to minimize the expected value of discounted cost over the entire future at the beginning of the enterprise. We shall consider first a sub-problem to this one: to minimize expected discounted cost subject to the constraint that stock on hand plus outstanding orders should not exceed a fixed limit and on the assumption that carrying cost is zero.

1.3. The main mathematical step is to find an expression for the discounted expected cost. Since all actions of inventory policy taken at a time t or after, affect the cost only after the lag period, i.e. at time $t + T$ or later, it is appropriate to maximize the expected discounted cost after T days, i.e. as of time $t + T$. What are the variables known at time T that the expected discounted cost at time $t + T$ depends on?

Because of the nature of the Poisson process future demand is at any time independent of the past. Future cost at time $t + T$ therefore depends on the availability of stocks at and after time $t + T$ only. Since demand was assumed postponable, the age composition of the orders outstanding at time t does not affect the availability of stocks at time $t + T$.* Therefore there is no reason why the policy at time t and afterwards, regarding stock availability at time $t + T$ and afterwards, should depend on the dating of the orders outstanding [illegible] may all be lumped together into a "net stock" as far as

* Namely an order τ days old at time t will mature into a physical stock at time $t + T - \tau$, so that at time $t + T$ all orders outstanding at time t, and only these, will have grown into stock.

their effect on stocks and the policies affecting stocks after T days are concerned. We conclude that expected discounted loss computed at time t for the time $t + T$ and therefore should be a function of net stock at time t only.

2. Analysis.

2.1. The mathematical argument of this section may be summarized as follows. Suppose that net stocks are at their maximal admissible level. As this is diminished through demand by one unit at a time a level of net stock will be reached ultimately, at which ordering cost is outbalanced by the advantages of an increased net stock. If the limit on net stock was an effective one, as we may assume, the net stock level will be raised by the order to its maximal level again. Net stock thus performs a cycle, and the ordering policy is reduced to two constants, the upper and lower levels of this cycle. The lower level may be termed a "safety allowance" and the difference of upper and lower level the "order size". The optimal value of safety allowance and order size may be found by minimizing expected discounted cost with respect to these two parameters, if a carrying cost is added proportional to safety allowance plus order size.

2.2. To fix ideas let us consider first the problem for unlagged delivery This is a case of a pure order size problem, because any shortage of stocks may be completely avoided by ordering immediately after the stock level has dropped to zero. The "safety allowance" aspect of the inventory problem has therefore disappeared. We denote by

t	time (in days as units)
$x(t)$	stock
λ	mean demand per day
K	fixed ordering cost plus an interest charge to allow for the deferred charging
k	proportional ordering cost (price) plus the corresponding interest charge
α	daily interest rate
b	stock limit
c	carrying cost per unit of the maximal amount stocked
$\ell(x)$	expected loss, a function of the stock level only
n	order size.in which there is

We consider first the case in which there is a bound on (net) stock and zero carrying cost. Let the stock be x and let $\ell(x)$ be given. After a sale of one unit the expected loss is either $\ell(x-1)$ or, if an order of n items is made, $\ell(x-1+n) + K + kn$. Define a function $\delta(n) = \begin{cases} 0 \\ 1 \end{cases}$ according as $n \begin{Bmatrix} = \\ > \end{Bmatrix} 0$. The ordering policy for a stock level $x - 1$ will be such as to minimize with respect to n $l(x + n - 1) + K\,\delta(n) + kn$, subject to the constraint that $n \leq b$. To obtain an expression for $\ell(x)$ consider its variation during a small time interval dt. With probability λdt one item will be demanded, and with probability $1 - \lambda dt$ no demand will arise. The expected loss dt units of time later is therefore

$$\min_{n \leq b} [\ell(x+n-1) + K\,\delta(n) + kn] \quad \text{and} \quad \ell(x)$$

respectively. Since no loss arises during dt, provided that $x > 0$, and since the discount factor for losses dt units of time later is $e^{-\alpha dt}$, the loss function satisfies an identity

$$\mathcal{L}(x) = e^{-\alpha dt}\left[(1 - \lambda dt)\,\mathcal{L}(x) + \lambda dt \min_{n \leq b} \left(\mathcal{L}(x+n-1) + K\delta(n) + kn\right)\right]$$

For small dt $e^{-\alpha dt} \approx 1 - \alpha dt$. An easy calculation now gives

$$\mathcal{L}(x) = \rho \min_{n \leq b} \left[\mathcal{L}(x + n - 1) + K\,\delta(n) + kn\right]$$

where $\rho = \dfrac{\lambda}{\alpha + \lambda} < 1.$

It follows in particular that

$$\mathcal{L}(x) \leq \rho\,\mathcal{L}(x - 1) < \mathcal{L}(x - 1)$$

In other words $\mathcal{L}(x)$ is a decreasing function of the stock level. And

$$\mathcal{L}(x + n) \leq \rho^n \mathcal{L}(x)$$

It follows that for

$$K + kn + \mathcal{L}(x + n) \leq \mathcal{L}(x)$$

it is sufficient that

$$\mathcal{L}(x + n) \leq \rho^n\,\mathcal{L}(x) \leq \mathcal{L}(x) - K - kn$$

Or [illegible]

$$\mathcal{L}(x) \leq \frac{K + kn}{1 - \rho^n}.$$

The largest value of the left hand side is $\chi(0)$, the smallest value of the right hand side is $\underset{n \leq b}{\text{Min}} \frac{K + k}{1 - \rho^n}$. The question is whether in

$$\chi(0) \leq \underset{n \leq b}{\text{Min}} \frac{K + kn}{1 - \rho^n} = \frac{K + k\bar{n}}{1 - \rho^{\bar{n}}}$$

the " = " sign can be attained. This is in fact the case if

$$\chi(\bar{n}) = \rho\,\chi(\bar{n} - 1) = \dots \rho^{\bar{n}}\,\chi(0) \quad \text{and} \quad \chi(0) = K + k\bar{n} + \chi(\bar{n})$$

as an easy calculation shows. These last equations hold when orders are made at stock level zero only and the order size is $\bar{n} \leq b$.

2.3. If carrying costs cx are introduced, proportional to the maximal stock, the optimal order size is determined by the minimization of

$$\frac{K + kn}{1 - \rho^n} + cn$$

subject to no side conditions. The first order conditions for $n = \bar{n}$ at minimum cost are

$$\Delta\left(\frac{K + kn}{1 - \rho^n} + cn\right)_{n=\bar{n}} \geq 0$$

$$\Delta\left(\frac{K + kn}{1 - \rho^{n-1}} + c(n-1)\right)_{n=\bar{n}} \leq 0$$

The first inequality becomes

$$-(K + kn)\ \frac{1-\rho}{1-\rho^n}\ \frac{\rho^n}{1-\rho^{n+1}} + \frac{k}{1-\rho^{n+1}} + c \geqq 0$$

$$-\ K - kn + \frac{k}{\rho^n}\ \frac{1-\rho^n}{1-\rho} \geqq -c\ \frac{1-\rho^n}{1-\rho}\left(\frac{1}{\rho^n} - \rho\right)$$

$$K + kn - k \sum_{j=1}^{n} \frac{1}{\rho^j} \leqq c \sum_{j=1}^{n}\left(\frac{1}{\rho^j} - \rho^j\right)$$

The second inequality is correspondingly

$$K + k(n-1) - k \sum_{j=1}^{n-1} \frac{1}{\rho^j} \geqq c \sum_{j=1}^{n-1}\left(\frac{1}{\rho^j} - \rho^j\right)$$

We may write for simplicity

$$(1) \qquad K + k \sum_{j=1}^{n}\left(1 - \frac{1}{\rho^j}\right) \approx c \sum_{j=1}^{n}\left(\frac{1}{\rho^j} - \rho^j\right)$$

The solution n of (1) is always a minimizer because the second differance of the minimand is always positive, when (1) is satisfied. That (1) can have only one solution given K, k, c, and e follows from the fact that

$$k \sum_{j=1}^{n}\left(\frac{1}{\rho^j} - 1\right) + c \sum_{j=1}^{n}\left(\frac{1}{\rho^j} - \rho^j\right)$$

is monotonically increasing.

2.4. Equation (1) above may be simplified considerably by noting that for ρ close to unity we may write $\rho^j = (1-\epsilon)^j \underset{=}{\sim} 1 - j\epsilon$. Substituting and simplifying one obtains

$$K \cong (2c + k)\,\alpha \sum_{j=1}^{n} j$$

$$n^2 + n - \frac{K(\alpha + \lambda)}{(\frac{k}{2} + c)\alpha} = 0$$

This approximation is valid if we choose our time unit so that $\lambda = 1$ unit per unit time and if the interest rate is small. The latter equation has one positive root given by

$$n = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4K(\alpha + \lambda)}{(\frac{k}{2} + c)\alpha}}$$

But since α is small, the daily rate being about 0.0002 if the annual rate is 0.06, this is essentially,

$$n \cong \sqrt{\frac{2\lambda K}{\alpha k + 2\alpha c}}$$

This is the "economic lot size" formula which is well known, (cf. Whitin [5, esp. p. 38]). It enters here as a close approximation for the case of uncertainty but no lag in orders. If we have the following values:

λ = 1 unit per day

α = 0.0002 per day

K = $1.80 per order

k = 0.30 per unit

c = $10,

then the optimum order size is determined to be about 78 units.

2.5. We now consider the inventory problem, with a positive lag but without carrying cost. Denote by

- T — the lag
- A — a "fixed" penalty per day of shortage independent of the amount of the shortage
- a — a penalty per day and item proportional to the amount of the shortage
- x — the level of stock plus outstanding orders
- $\ell(x)$ — the expected loss starting T days hence, given that the current level of stock and orders is x

With a probability $p(y) = \frac{(\lambda T)^y}{y!} e^{-\lambda T}$ an amount y will be demanded during the period T. The probability of a shortage during the time interval T, $T + dt$ is therefore $\sum_{y=x+1}^{\infty} p(y)$ and the expected discounted penalty

$$e^{-\alpha T} \left[A \sum_{y=x+1}^{\infty} p(y) + a \sum_{y=x+1}^{\infty} (y - x)\, p(y) \right] dt$$

Using the reasoning of 1.4 the expected loss at time t may be expressed in terms of the expected loss at time $t + dt$ as follows

$$\ell(x) = e^{-\alpha T} \left[A \sum_{x+1}^{\infty} p(y) + a \sum_{x+1}^{\infty} (y - x)\, p(y) \right] dt$$

$$+ e^{-\alpha dt} (1 - \lambda dt)\, \ell(x) + e^{-\alpha dt} \lambda dt \cdot \underset{n \geqq 0}{\text{Min}} \left[l(x-1+n) + K\,\delta(n) + kn \right]$$

As $dt \to 0$ we obtain the recursive relation

$$\ell(x) = \rho \underset{n}{\text{Min}} \left[\ell(x - 1 + n) + K\,\delta(n) + kn \right] + h(x)$$

where $\quad \rho = \frac{\lambda}{\alpha+\lambda}$

$$h(x) = \frac{e^{-\alpha T}}{\alpha+\lambda}\left[A \sum_{x+1}^{\infty} p(y) + a \sum_{x+1}^{\infty} (y-x)\, p(y)\right]$$

If this equation has a solution $\ell(x)$ this means that there exists a policy which reduces the expected loss to a function of the "net stock" x only. Let $H = \underset{n \geq 0}{\text{Min}} \, [\ell(n) + kn] = \ell(x^*) + kx^*$. This minimum must exist because $\ell(0)$ is finite and since $\ell(x) \geq 0$, $\ell(n) + kn$ increases indefinitely as $n \to \infty$. Then if $x - 1 \leq x^*$

$$\underset{n \geq 0}{\text{Min}} \, [\ell(x-1+n) + kn] = H - k(x-1)$$

Therefore

$$\underset{n \geq 0}{\text{Min}} \, [\ell(x-1+n) + K\,\delta(n) + kn] = \begin{cases} H + k - k(x-1) \\ \ell(x-1) \end{cases}$$

$$\text{according as} \quad \ell(x-1) + k(x-1) \begin{Bmatrix} > \\ \leq \end{Bmatrix} H + K$$

Now $\ell(x)$ is by definition a non-increasing function. For otherwise the expected loss could be decreased by disposing of some stock, which cannot happen in the absence of carrying cost. $\ell(x) + kx$ is therefore strictly decreasing. This means that for all $x - 1$ not below some critical value, s, say, the second alternative holds. Therefore the equation gives rise to the system

$$(1) \qquad \begin{aligned} \ell(x) &= \rho\, \ell(x-1) + h(x) \qquad\qquad x > s \\ \ell(s) &= H + K - ks = \ell(x^*) + K + k(x^* - s) \end{aligned}$$

2.6. Suppose now that an upper bound S is imposed on the stock level, (because of capital rationing or simply because an optimal value of this parameter has been chosen in advance.) Then since $f(x)$ is non-increasing x^* is replaced by S as we may assume that $S < x^*$. The system (1) now follows is with S in place of x^*. Solving for $f(S)$

$$fS = \rho^n f(s) + \sum_{j=1}^{n} h(s + j) \rho^{n-j}$$

$$f(s) = f(S) + K + kn$$

where we have put $n = S - s$. n is the order size. We obtain

$$f(S) = \frac{1}{1-\rho^n} \left[\rho^n (K + kn) + \sum_{j=0}^{n-1} h(S - j) \rho^j \right]$$

If carrying cost over all future is now introduced and assumed proportional to the maximal net stock S, $= cS$, then the total expected loss at time zero calculated from T days or afterwards is

$$\frac{1}{1-\rho^n} \left[\rho^n (K + kn) + \sum_{j=0}^{n-1} h(S - j) \rho^j \right] + cS$$

and this is to be minimized with respect to both S and n. (Instead of S we could also have chosen s to be the independent variable, without changing the solution).

2.7. The penalty function written explicitly is

$$h(x) = \frac{Ae^{-\alpha T}}{\alpha + \lambda} \sum_{j=x+1}^{\infty} \frac{e^{-\lambda T}(\lambda T)^y}{y!} + \frac{a\lambda Te^{-\alpha T}}{\alpha + \lambda} \sum_{y=x+1}^{\infty} (y-x) \frac{e^{-\lambda T}(\lambda T)^y}{y!}$$

$$= \frac{(A+a\lambda T)}{\alpha + \lambda} e^{-\alpha T} \sum_{y=x+1}^{\infty} \frac{e^{-\lambda T}(\lambda T)^y}{y!} + \frac{a\lambda T}{\alpha+\lambda} e^{-\alpha T} \frac{e^{-\lambda T}(\lambda T)^x}{x!}$$

$$- \frac{ae^{-\alpha T}}{\alpha + \lambda} x \sum_{y=x+1}^{\infty} \frac{e^{-\lambda T}(\lambda T)^y}{y!}$$

If we write $\beta_1 = \dfrac{(A + a\lambda T)e^{-\alpha T}}{\alpha + \lambda}$, $\beta_2 = \dfrac{a\lambda T e^{-\alpha T}}{\alpha + \lambda}$

$$P(x) = \sum_{y=x+1}^{\infty} \frac{e^{-\lambda T}(\lambda T)^y}{y!} = \Pr\left\{\text{DEMAND IN REPLACEMENT PERIOD EXCEEDS } x\right\}.$$

$$p(x) = \frac{e^{-\lambda T}(\lambda T)^x}{x!} = \Pr\left\{\text{DEMAND IN REPLACEMENT PERIOD EQUALS } x\right\}.$$

then we have

$$h(x) = \beta_1 P(x) + \beta_2 p(x) - \frac{\beta_2}{\lambda T} x\, P(x)$$

We want to find the minimizers $\bar{n}$ and $\bar{S}$ of

$$\frac{\rho^n}{1-\rho^n}[K + kn] + \frac{1}{1-\rho^n} \sum_{j=0}^{n-1} \rho^j \left\{\beta_1 P(S-j) + \beta_2 p(S-j) - \frac{\beta_2}{\lambda T}(S-j) P(S-j)\right\} + cS$$

In order that $\bar{n}$ be the minimizer with respect to n of the function $f(s,n)$ whose arguments are discrete it is necessary that: $\Delta_{\bar{n}} f \geq 0$ and $\Delta_{\bar{n}-1} f \leq 0$ and similarly for S. The approximate values of S and n which will minimize (7) must therefore satisfy

$$\Delta_S f(\bar{S},\bar{n}) \cong 0 \quad \text{and} \quad \Delta_n f(\bar{S},\bar{n}) \cong 0.$$

Taking differences one finds

(8) $$[\beta_1 - \beta_2] \sum_{j=0}^{\bar{n}-1} \rho^j p(\bar{S}-j) + \beta_2 \sum_{j=0}^{\bar{n}-1} \rho^j p(\bar{S}-j-1)$$

$$- \frac{\beta_2}{\lambda T} \sum_{j=0}^{\bar{n}-1} \rho^j \left\{ (\bar{S}-j)\, p(\bar{S}-j) - P\,(\bar{S} - k - 1) \right\} \approx \bar{S}(1 - \rho^{\bar{n}})$$

(9) $$[K + kn] + \beta_1 \sum_{j=0}^{\bar{n}-1} \rho^j P(\bar{S}-j) + \beta_2 \sum_{j=0}^{\bar{n}-1} \rho^j p(\bar{S}-j)$$

$$- \frac{\beta_2}{\lambda T} \sum_{j=0}^{\bar{n}-1} \rho^j (\bar{S}-j)\, P\,(S-j) \approx$$

$$\left[h\rho + \beta_1 P(\bar{S} - \bar{n}) + \beta_2\, p(\bar{S}-\bar{n}) - \frac{\beta_2}{\lambda T} (\bar{S}-\bar{n})\, P(\bar{S}-\bar{n}) \right] \sum_{j=0}^{\bar{n}-1} -\rho^j$$

In the special case where $a = 0,\ \beta_2 = 0$ (8) and (9) become:

(8') $$\sum_{j=0}^{\bar{n}-1} \rho^j p(\bar{S}-j) = \frac{\bar{S}(1-\rho^{\bar{n}})}{\beta_1}$$

(9') $$P\,(\bar{S}-\bar{n}) \sum_{j=0}^{\bar{n}-1} \rho^j - \sum_{j=0}^{\bar{n}-1} \rho^j P(\bar{S}-j) = \frac{K}{\beta_1} + \frac{k}{\beta_1} \left[\bar{n} - \rho \sum_{j=0}^{\bar{n}-1} \rho^j \right].$$

In the next section we shall discuss a numerical example and computational procedure.

2.7. In interpreting the equations it is useful to refer them to a natural time unit $\frac{1}{\lambda}$ which equals the mean time distance between demands. $A^* = \frac{A}{\lambda}$ measures the adjusted time rate of penalty. Since α is the interest per day $\rho = \frac{1}{1 + \frac{\alpha}{\lambda}}$ is the inverse of the interest factor for this time unit, that is, its discount factor. Events that are spaced one time unit apart are discounted by this factor ρ.

Consider now a "turn over period" during which net stock runs through all its levels S, S-1, ...S-n+1. What is the probability that actual stocks during a "basic period" T days removed from this turnover period will reach the zero level? p(S-k) is the probability of exactly S-k sales during the time T (in old units) following the time interval in the turnover period during which the net stock level is S-k. Since the events of stocks reaching the zero level during correspondent time intervals in the basic period are mutually exclusive, the probability of stocks running down to zero during an entire basic period is $\sum_{j=0}^{n-1} p(\bar{S}-j)$. Introducing the proper discount factors ρ^{j+1} associated with the events on the assumption that they will happen at the end of the respective intervals we obtain the probability sum in (8'). To allow for the fact that events are still T days off a further factor $e^{-\alpha T}$ is introduced. If the same events are considered one period after the one considered, their probabilities must be discounted by ρ^n, this being the discount factor for the average time it takes to complete n sales. Summing over all periods gives rise to a factor $\frac{1}{1-\rho^n}$. In summary we have a discounted probability expression

$$\frac{e^{-\alpha T}}{1-\rho^n} \sum_{j=0}^{\bar{n}-1} \rho^{j+1} p(\bar{S}-j) = \bar{p}, \text{ say.}$$

Equation (8') shows this to be equal to the price ratio

$$\frac{c}{A^*} \text{ where } A^* = \frac{A}{\lambda} \text{ is the adjusted penalty.}$$

This is the price ratio of inventory and penalty. According to the theory of resource allocation an optimal balance of two factors requires that their price ratio should equal the marginal rate of substitution of one for the other. Thus $\bar{p}$, the discounted probability of stock depletion in the various basic periods, is seen to represent the increment in chances for incurring penalties due to a unit decrease in inventory.

In order to interpret (9') we bring it into the form

$$\frac{e^{-\alpha T}}{1-\rho^{\bar{n}}} \sum_{j=0}^{\bar{n}-1} \rho^{j+1} [P(S-\bar{n}) - P(S-j)] = \frac{\frac{K}{1-\rho^{\bar{n}}} + \rho k}{A^*}$$

On the right hand side is again a price ratio: $\frac{K}{1-\rho^{\bar{n}}}$ is the total discounted outlay for fixed ordering cost, given an order size n; $\frac{\rho k}{1-\rho}$ is the outlay for proportional ordering cost if this were charged after every sale. These two terms correspond roughly to the cost of an order size $\bar{n}$. A^* again denotes the penalty per average time interval between sales.

What does the expression in brackets on the left hand side mean? $P(\bar{S}-\bar{n})$ is the probability of selling at least $\bar{S}-\bar{n}$ units during the lag period of T days $P(\bar{S}-\bar{n}) - P(\bar{S}-\bar{k})$ is therefore the probability of selling not less than S-n and not more than S-k-1. This is therefore the probability of depletion during a basic period (a period following by T days a complete cycle of net stocks) after the kth interval. The left hand side represents the discounted sum of probabilities of running out of stock during basic periods after various dates measured in "natural" time units.

The equation thus tells us that the discounted sum of depletion probabilities after the various dates is the increment in discounted depletion probability due to unit increase in order size.

3. Computation, an Example:

3.1. The above equations may be simplified for computation by noting that if α_1 the rate of interest, is small if we choose our time units so that $\lambda = 1$ unit per unit time ρ is close to one and $\rho^j = (1 - \epsilon)^j \cong (1 - j\epsilon)$. Substituting in (8') and (9') we have

$$(8'') \qquad \sum_{j=0}^{n-1} \rho^j \, p(\bar{S}-j) \cong \left(\frac{c}{\beta_1} \epsilon \right) \bar{n}$$

$$(9'') \qquad P(\bar{S}-\bar{n}) \sum_{j=0}^{\bar{n}-1} \rho^j - \sum_{j=0}^{\bar{n}-1} \rho^j \, P(\bar{S}-k) \cong \frac{K}{\beta_1} + \left(\frac{k}{\beta_1} \epsilon \right) \bar{n}$$

The equations are now in a form which is fairly convenient for calculation. The right hand side of each depends only upon $\bar{n}$ while the left hand sides depend upon both $\bar{S}$ and $\bar{n}$. Values of $P(\underset{\mu}{h})$ and $p(\underset{\mu}{h})$ have been tabulated by Molina [4]. Given λ and η, values of ρ^j and $\sum_{j=0}^{n-1} \rho^j$ can easily be tabulated. Using these, cumulative multiplication yields the sums indicated on the left hand sides of these two equations. Hence one can, for selected values of S, calculate those values of n which satisfy equations (8") and (9") respectively. By repeated calculations one can find those values of S and n which satisfy both these equations. The process is simplified here since S and n take integral values only.

To illustrate we shall present a numerical example based upon the stocking of machine repair parts by a manufacturing firm we have studied. We assume the following values of the parameters:

λ = 1 unit per day

T = 90 days

α = 0.0002 (corresponding to an annual rate of interest of approximately 6 percent)

B = \$1.80

b = \$0.30

c = \$10 (\$0.60 per year)

A = \$1.80 per day

a = 0

It will be remembered that c is the "capitalized" carrying cost per unit. Figures 1A and 1B illustrate graphically the solution for n for fixed S = 130 of equations (8") and (9"). The left hand sides of each are convex* functions of n in the relevant range. The right hand side of (8") is proportional to n while the right hand side of (9") is approximately linear in n, and almost constant. The intersections give the appropriate values of n for S = 130. Figure 2 shows two lines. The steeper is the locus of values of S and n which satisfy equation (8"), the flatter the locus of those satisfying equation (9"). For this example these are approximately linear. Their intersection gives $\bar{S}$ and $\bar{n}$, in the example 140 and 33 respectively. It is interesting to note that, on the average, about three orders will be placed during a ninety day period so on the average about two orders are outstanding when a new order is placed. Here S = 107 and since mean demand during the replenishment period is 90 units the "safety margin" is 17 units. Referring to Molina's tables one finds that the probability of a backlog occurring during the replacement period, that is that demand will exceed 107 units during the period, the loss is minimized by carrying such a level of stock so as to run out on the average of once in thirty times. It will be recalled that for the unlagged case, section 1.4 above, the optimal order

* That is, they increase at an increasing rate.

size, and maximum stock level, was 78 units. Thus the introduction of a "long" lag and a backlog penalty reduces order size by about a half and almost doubles the maximum stock level.

3.2. An examination of figures 1A and 1B reveal two interesting points. Figure 1B shows that the parameter b has little effect upon the order size since the slope of the line $\frac{K}{\beta_1} + (\frac{k}{\beta_1}\epsilon)\, n$ is negligible. This is because of the fact that one must pay the proportionate part of the ordering cost regardless of how often one orders. If 30 units are ordered now rather than 15 units now and 15 units a month hence, the only difference in the proportionate cost incurred enters through the discounting of future outlays and this effect is small. One also notes that, for $S = 130$, after $n = 25$ the sum of the discounted probabilities represented by the convex curve increases very rapidly. Thus, after a point, an increase in fixed ordering cost or a decrease in the penalty for backlog has little effect upon the order size. Also, in both cases these curves become quite flat for n less than 15 which indicates that after a point a decrease in the fixed ordering cost or an increase in the penalty for shortage causes the order size to decrease rapidly toward zero, for any S.

Using curves like figures 1 and 2 the effects of changes in cost parameters are readily shown. Suppose first that carrying cost c is increased. Then the slope of the right hand side of (8") is increased and figure 1A tells us that for any given upper stock level S, the optimal order size n increases. This means that the curve showing the locus of values of S and n satisfying equation (8") in figure 2 must shift upward or that $\bar{S}$ and $\bar{n}$ must decrease. Thus increased carrying charges mean a smaller maximum stock level. The effects of an increase in the fixed ordering cost K can be shown in a similar fashion.

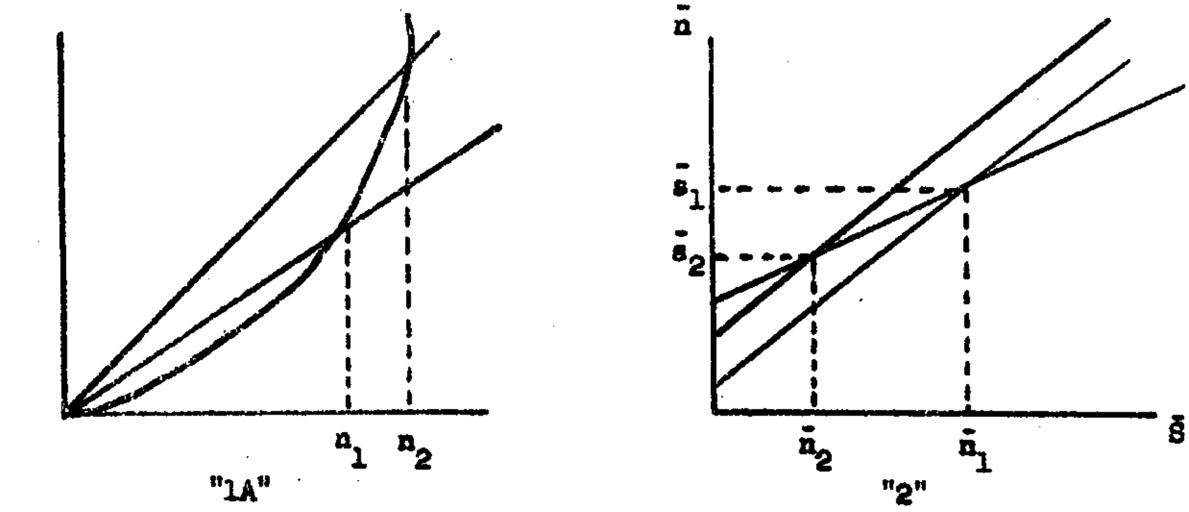

Fig. 3

For given S, n increases so that in figure 2, the curve showing the locus of values of S and n satisfying equation (9") shifts upward. This increases both $\bar{S}$ and $\bar{n}$. Finally, let the fixed penalty A decrease. Then both curves of figure 2 are shifted up and n must increase. The effect on $\bar{S}$ is not readily apparent but presumably $\bar{S} - \bar{n} = \bar{s}$ decreases. In all cases the ascertainable results of changes in the parameters on the solution are in agreement with economic intuition.

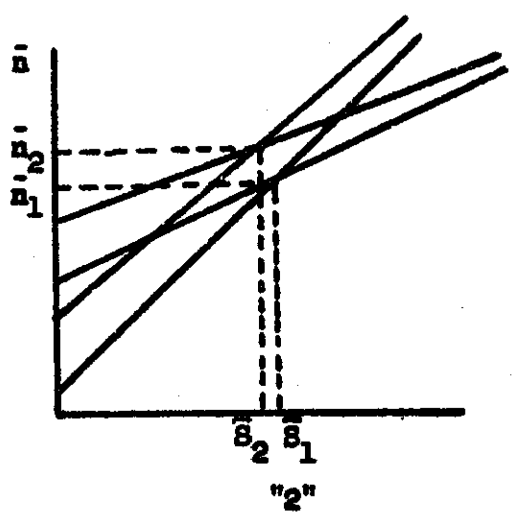

Fig. 4

4. Conclusion: Safety Allowances Reconsidered

As a rule of thumb it is sometimes suggested, that the safety allowance, the amount on hand when an order is placed, should equal the mean demand during the time that must elapse before delivery. The fact, that a shortage is so much costlier than an excess of stock would seem to require that the safety allowance should actually be larger than the mean demand. The amount by which it should be larger in turn will be a function of the order size. This point, that order size and safety allowance are jointly dependent variables, has been brought out by forcefully by Whitin [5, pp. 56-62].

Here we want to emphasize another fact about safety allowances. The tacit assumption in the arguments of Whitin and the earlier writers criticized by him, is that not more than one order at a time is ever outstanding. If delivery lags are considerable this forces up the order size to at least the level of the mean demand in the lag period, as we have seen. The preceding analysis, which is free from this assumption, shows that the ordering point should not be given in terms of stock on hand alone, but in terms of the sum of stocks and all orders outstanding. (If the assumptions about postponable demand and a constant probability of one demand at a time are not satisfied, no ordering point exists, but instead there is a more complex "ordering function" which must involve the dates of previous unfilled orders). When delivery lags are long and the value, the proportional ordering cost, of the commodity is high, our equations show that it becomes optimal to stagger orders: the difference between the upper and lower ordering point may be less than the mean demand during the lag period. This seems a rather interesting conclusion about the nature of an optimal ordering policy in the presence of delivery lags.

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