COWLES COMMISSION DISCUSSION PAPER: ECONOMICS NO. 2122

Some Aspects of the Airline Reservations Problem*

Martin Beckmann**

with the assistance of F. Bobkoski

June 9, 1955

---

* Research undertaken by the Cowles Commission for Research in Economics under contract Nonr-358(01), NR 047-006 with the Office of Naval Research.

** I am indebted to the following airline officials who have patiently answered by many questions, Messrs. E. C. McQuillen, P. Moore and R. Lets, Mr. N. Syberg, Messrs. Bush, R. F. Burkhardt, and C. Lennarts. My special thanks is due to the airline which has supplied the data for the histograms. Mr. Bobkoski has carried out all statistical estimation and most of the calculations. Mr. R. Muth has helped with the calculation of examples. I have benefitted from comments by J. Marschak.

1. Introduction

The selling of a fixed quantity of a perishable commodity by many agents offers a good example for decision making and communication in teams. The economically most significant cases occur in the field of transportation, where a limited number of seats on a means of transportation is available and being offered for sale in advance on a reservation basis.[1] This occurs on the largest scale in airline transportation. Naturally the systems adopted must be understood in the light of the characteristics of the demand for airline transportation. This paper studies some aspects of the present airline reservation system from the team theoretical point of view [Section 2]. It also explores the distribution pattern of the various elements of demand, such as of manifested passengers with reservations, of stand by passengers without reservations, and of no shows and other losses. Fairly good fits were obtained with Gamma distributions, a class of distribution functions that can be justified also on theoretical grounds [Section 3]. This permits one to take a fresh look at the problem of the proper sales limit for a flight of known demand characteristics [Section 4]. A few calculations have also been made which may help to

---

1 The rationale of the reservations system as against the accomodation of a varying demand to a rigid supply through a flexible pricing system has been thus summarized in a passage of a recent paper by W. S. Vickrey: "...Most customers make fairly firm plans considerably in advance of the time the service is actually rendered. Even where it might be technically possible to fix rates at the last minute, as for example by auctioning off the available space to the highest bidders just before departure, this would fail to furnish an adequate guide for the planning of the customers, inasmuch as they would not in most cases be expected to be able to make as good estimates of the probability distribution of the ultimate price, as of some advance date, as could the operators of the service. Moreover, even if such estimates could be drawn up and effectively communicated without undue expense, most customers would find it unsatisfactory to be placed in the position of having to commit themselves to the use of a service the cost of which is uncertain over a wide range. It is thus essential to provide the customer at the time he makes his plans with some fairly firm price which will serve both as a guide in planning and as a hedge against having to make what may prove an unexpectedly and undesirably large expenditure." [4]

relate the concepts used in this paper to more familiar notions such as the load factor, and which show how to operate with these distribution functions. [Section 5].

2. Description of Reservation Systems

2.1. Sell and Record

The reservation rules practiced today are of great interest to students of teams since they are the outcome of many years' experience and considerable experimentation. It is not unlikely that the current reservations practice will again undergo some change in the near future: On routes of high density of scheduled flights already many passengers tend to travel without arranging for reservations. It is conceivable that on such principal routes the reservations system may be abolished entirely although some arrangement for priority may still be necessary. A certain aspect of the communication program may be changed moreover by the introduction of electronic recording devices. However these changes will come about slowly, if at all. The present state is thus by no means of just academic interest.

In the following the words sales and reservations will be used interchangably. Sales are made by so called reservation agents, the individuals who answer inquiries by telephone. The responsibility for keeping sales within the limits of the flight's capacity rests with offices called space control. A reservations system is a set of communication rules between the reservation agents and space control, -- and finally between space control and the airport agents at the departure stations.

The system adopted at present by all major airlines (with minor modifications) is that of "sell and record". Under this system every agent may sell space on any flight (though not more than a certain maximum number of seats at one time) unless he has previously received a message from space control stopping sales to that flight. In this case the agent may put the passenger on a waiting list if he so desires. Ordinarily each sale and each cancellation are reported immediately to space control. When the accumulated sales have reached a certain level, space control issues a "stop sales" message to all reservation agents. If during the time between the sending of the sales messages and the receipt of a stop sales message, sales have taken place to an extent exceeding a certain critical level, "protect sales" messages are sent by space control. An effort is then made by reservation agents to clarify the status of any passengers, who by failing to purchase a ticket or otherwise have indicated that they are potential "no shows". Any cancellations thus obtained are reported to space control until space control sends a message putting this flight back on an ordinary "stop sales" basis.

Passengers on waiting lists are not reported to space control, but a message is sent to the effect that a waiting list is being kept. Space made free by cancellations is filled up from the local waiting list. If the waiting list has been emptied, space control is advised about this only when unused cancellations reach a considerable level. If space control wants to allocate unused cancellations it sends a "report cancellations" message. It then receives all cancellations and reservation requests and keeps a waiting list centrally. But this is not usual. Any unused part of capacity that is available when the flight goes on stop sale is allocated to the origin station of the flight and used by this station as any space freed by cancellations.

A few hours before flight departure control over the flight is handed from space control to "departure control" at the airport station. This is done by sending a passenger list, a manifest, to the airport agent. The passengers actually carried on the flight differ from those manifested by various last minute changes.

	late cancellations		
losses	no shows	=	passengers with valid reservations failing to appear at flight departure
losses	misconnections	=	passengers from other flights that are late
adds	errors	=	passengers showing up with valid reservations which, by mistake were not recorded
adds	stand bys	=	passengers without reservations waiting at the airport
	removals	=	passengers with valid reservations refused transportation because of oversales.

After departure of the flight a corrected manifest is sent to the airport agent at the first stop, who on the basis of this information can allocate space to stand bys, etc.

If the flight is nonstop, space control rests with the reservations office at the station of departure. For multistop flights space control is either centralized in one office for the entire system or in one office for each "region".

Various refinements to this basic system are in use. Since sales do not reach a large volume (ordinarily) before approximately one week prior to departure date, reports of sales are often not required until either a certain date or until a certain allotment has been reached. Alternatively sales will be reported in blocks of, say, five only until a "post advisory message" is received from space control. Thereupon all sales will be reported singly. Sometimes stations originating very few passengers to a flight are not included in the stop sales circuit and may either sell and record all the time, or must "request and confirm" any reservation to the flight in question.

Usually space control sends a "flight check message" to the reservation offices of the various originating stations listing all the reservations received from that office for a given flight, in order to spot any errors. This is necessary only when the flight was "posted" i.e., was in a "stop sales" stage.

2.2. In order to see more clearly the rationale of this communications system, we shall attempt a formal description in terms of Marschak's model of a team.

2.2.1. Space control and the reservation and airport agents form a team with a common goal. Presumably this goal is the maximization of the airline's profits. From the emphasis that is put on the load factor, one may get the impression that this goal is the maximization of the load factor subject to the constraint that oversales stay within prescribed limits, but regardless of communications cost. A closer inspection of the communication rules which are clearly designed to keep communications cost low, dispells the last opinion at least with regard to the long run.

The "sell and record" system is eminently suitable to rule out any particularistic interest of reservations offices. Under the former system of space allotments to various stations, the latter often felt a proprietary interest in their allotments and were reluctant to relinquish part of their quota to other stations which had exhausted theirs since this would jeopardize their own sales record.

Often reservation agents of one airline can "sell and record" space on flights of another airline. In this manner the team may even include members of other firms. Naturally if a choice exists the agents will try to sell space on a flight of their own airline. But in finding connections with flights on routes not served by this airline, the object of maximizing profit (through maximization of the chance of a sale) is best served by selecting the best available flight indiscriminately. In the reservations procedure the agent subjects himself to the rules of the other airline involved so that (in this context) he is a member of that team. The goal of the team was designated as maximal profit to the airline. This is to be understood as long run profit. In order to translate long run profit into the short run, a certain penalty must be attached to actions that tend to decrease long run revenue such as "protect sales" and removals of passengers. The profit to be considered by the "traffic department" (including reservations, space and departure control) is before operating cost, the latter being regarded as fixed with the given schedule. Thus the proper short run profit to be considered is passenger revenue -- minus communications cost -- minus penalties. Suppose for a moment that communication was costless and without delay so that every team member had all the information that is useful to him. What would be the best decision rules to follow?

Obviously demand for reservations would be accepted until the flight is sold to capacity. After this passengers would be accepted on waiting lists and would be given reservations, provided they still want them, in the order that cancellations come in. Waiting lists would be closed at a certain level. Oversales are impossible. But under-utilization of capacity will occur because of no shows, late cancellations and misconnections. The question arises whether it would not be profitable to book a few more passengers than the capacity of the flight allows in order to compensate for the anticipated lost passengers. This decision problem which will be considered further in section 4, arises regardless of the reservations system used. It is also independent of the team aspects of the reservation problem.

It may be thought that communications cost is of such small order compared with the high price of unused seats on a flight that perfect communication should be attempted even though it costs a little. This is the rationale of the "request and confirm" system. Its principal defect is that it disregards the delay necessary in confirming a reservation to a passenger. This delay gives rise to a cost which is considerable: the probability of a loss of a passenger because of his unwillingness to wait. Under competitive conditions this cost is regarded to be prohibitive. The present sell and record system arose from a recognition of this cost element.

There are however, situations (encountered by certain European airlines) in which the moneycost of communication alone is large enough to encourage some degree of decentralization. Recourse is then had to some form of the ancient quota system. We shall not go into this here, although the problem of optimal allocation of quotas has some interesting aspects. Its solution is by no means as simple as it is sometimes assumed, for the optimal quotas need not correspond to the average demands.

2.2.3. Turning back to sell and record, the structural and functional aspects must be distinguished, or (in Marschak's terminology) the network and the communication rules.

The network is defined by the channels of communication that are open between the various team members. In the present context the network is specified by the location of space control for a given flight and be the set of agents "in the circuit", i.e. that receive stop sales messages.

2.2.4. Perhaps the most important elements of the communication rules are the degree of priority given to stop sales messages and the size of the "cushions". The cushion is the safety margin between the level at which a stop sales message is sent and the capacity of the flight. Stop sales messages rank in priority immediately after emergency, weather and operation messages. The delay is usually a matter of minutes. (5 minutes is a figure given by Grossman). Delays of the sales report messages may be more serious. However, the sales during the interim period are a very low fraction of total sales, according to one airline official well below one per cent. On a number of flights the cushion is in fact zero.

No delay between sales, reports and stop sales is incurred if space control is exercized by an electronic device such as the "reservisor". Here all reservation agents selling particular flights can ascertain the availability of seats on a flight immediately by operating a keyboard that is connected by wires with the space control memory drum. No cushion is required. System-wide reservisors have not been used as yet, but may be expected to find application in the near future.

The optimal size of the cushions in simple sell and record poses an interesting problem which will be further investigated in section 5.

Another fact of the communication rules that merits attention are the rules about the sending of messages after the flight is posted, i.e. when it is in the stop sales stage. A "wait list" message is superior to a "request and confirm" message (sometimes called a listing message) because "unless it has available the requested space, and, unless the control station sends a message of confirmation, the selling station need not telephone the passenger". (Grossman, p. 134).

2.3.5. We must turn briefly to the cost of communication, although little relevant information exists on this subject.

The cost of incoming calls for reservations consists in switchboard operators' and reservation agents' time. The average number of calls per agent per hour during the busiest hour of the day according to a 1947 estimate (Grossman, p. 166), is eighteen. From this an average length of 3 minutes per call has been estimated on the assumption that the agent is idle for 6 minutes per hour. Of course, the proper number of agents to be kept on duty is itself a decision variable to be determined by balancing the losses of potential customers through delayed answers against salaries. The marginal cost of a call is therefore an opportunity cost of losing calls from other customers which in the long run equals the salary of agents prorated on calls. The communication rules affect this cost via the length of calls. Sell and record in fact minimizes it. The average cost per long distance communication from a reservation agent to space control (or vice versa) has been given as 4 cents per message by an official of one airline.

An important part of communication cost is that of storing and processing information. The most conspicuous instance of the former is the keeping of "space advisory boards" in large reservations centers which list the status of every flight for every leg., i.e., whether it is on stop sale. Space control must also keep records of the sales reports received so as to be able to issue the necessary instructions. The marginal cost of space advisory boards and reservation records cannot even be guessed at.

3. Analysis of Demand Distributions

3.1. For present purposes one may distinguish three kinds of factors affecting the demand for transportation on a given flight:

- permanent factors: such as the fare and schedule of this flight, of competing flights, and of competing means of transportation; also the fares and schedules of connecting flights,
- periodic factors: the day of the week and the season, special holidays
- random factors: the weather and such events as conventions which enhance or depress demand

In the following analysis all permanent factors are held constant as well as possible. Period factors are lumped together with the random factors. Ideally, demand in different periods (days) should be stationary and independent, i.e. not influenced by what the actual demand was in preceding periods. For a given flight demand is then a random variable subject to a fixed distribution. Our object is to identify and interpret empirically observed frequency distributions of demand. These will then be used in calculations concerning certain decision problems of the reservations department.

Before surveying possible candidates among the elementary distribution functions, some obvious difficulties on the empirical side should be pointed out. The permanent factors are rarely ever completely constant, changes of schedule and of equipment occurring every few months. Until fairly recently total demand for airline transportation was subject also to a substantial upward trend. On flights with a heavy demand relative to capacity -- the most interesting ones as regards the decision problem -- only part of the demand distribution is observable, the frequency of demands for more seats than capacity appearing in the aggregate only.

3.2 The Demand for Reservations

3.2.1 The Poisson Approximation

Flying to a particular destination on a particular day may be regarded a "rare event" for any given passenger. The probability distribution of the total demand for a flight may therefore be assumed to be related to the Poisson distribution for rare events. More precisely we may argue as follows: let the time period during which reservations may be made for a flight be divided into a number of intervals such that the mean number of requests for reservations received during any one interval is the same. Let time now be measured in units of length proportional to the lengths of these intervals, i.e., in general different at different times. In terms of this redefined time concept consider the probability of receiving a call for a reservation during a small time interval. If the population of potential customers is large (consequently if the probability of any one desiring to go on this flight is small) then the number of reservations already received does not affect the probability of getting a further request for a reservation. Suppose, for simplicity that reservation requests are made for one seat at a time. If the interval considered is sufficiently small, the probability of getting one reservation request is proportional to the length of the interval, and the probability of getting more than one request is zero. It is well known [Feller, pp. 115-118] that on these assumptions the accumulated reservations over any fixed time interval of finite length must then be Poisson distributed. Let n be the number of reservations. Then the probability $f(n)$ of n reservations is

$$(3.1) \qquad f(n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

where $\lambda > 0$ is a constant, the mean of n. In particular this must be true for the total reservation requests that have accumulated at flight departure time, if we count as requests only those that are not cancelled again.

Examination of data restricted to seasons and to such weekdays that have an approximately stationary demand shows that there is some tendency for the empirical distributions to approximate the Poisson distribution, although the fit is not very good. (fig. 1,2) The usefulness of this approximation is restricted also because one would not wish to distinguish too many different cases of weekdays and seasons as would be required in order to arrive at a "pure" situation.

3.2.2. Poisson process with clustered demand

Histograms of passengers manifested seem to show that on certain flights demand occurs more frequently for an even number of seats than for an odd one. (fig. 3) This fact may be explained by assuming that requests for reservations obey a Poisson distribution, as before, but that each request is for either one or two seats, with constant probabilities. Assuming that p is the probability that one seat is requested and $q = 1 - p$ the probability of a request for two seats the probability of a total demand for n reservations can be shown to be

$$(3.2) \qquad p(n) = e^{-\lambda} \sum_{k=n-[\frac{n}{2}]}^{n} \frac{\lambda^k \, q^{n-k} \, p^{2k-n}}{(n-k)!\,(2k-n)!}$$

where $[a]$ denotes the largest integer not greater than a. $n - [\frac{n}{2}]$ is therefore the smallest integer greater than or equal to $\frac{n}{2}$. The number of terms in the expression for $p(n)$ increases by one when n increases by one from an odd to an even number and does not change when n increases from an odd to the next even number. This indicates a tendency for even numbered demands to have a larger probability.

Formula (3.2) is cumbersome. One expects that a more symmetric expression can be obtained on the assumption that demand occurs also for more than 2 seats at a time and that the clustering is subject to some simple probability distribution. Among plausible candidates the geometric distribution is perhaps the simplest one. In order to achieve a rapid decrease of the probability of larger clusters a rather considerable probability would be required for zero demands. It is natural now to associate the probabilities of the ordinary geometric distribution with demands that are one unit larger. Then

$$(3.3)\qquad \begin{aligned} p(0) &= 0 \\ p(k) &= (1-p)\,p^{k-1} \qquad k > 0 \end{aligned}$$

The probability of a total demand for n seats may now be derived as follows. The generating function of the Poisson distribution is $e^{-\lambda+\lambda s}$, that of the ordinary geometric distribution is $\frac{1-p}{1-ps}$ and that of the geometric distribution as used here is $\frac{(1-p)s}{1-ps}$. We are interested in sums of geometrically distributed variables for which the number of terms obeys a Poisson distribution. The generating function of the distribution of these sums is obtained by substituting in $e^{-\lambda+\lambda s}$ the expression $\frac{1-p}{1-ps}$ for s [Feller, p. 223]. Thus

$$\phi(s) = e^{-\lambda + \lambda\frac{(1-p)s}{1-ps}}$$

is the desired generating function. This may be expanded as follows.

$$\phi(s) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k s^k}{k!} \left(\frac{1-ps}{1-p}\right)^{-k}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \sum_{i=0}^{\infty} \binom{-k}{i} (-p)^i s^{i+k}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \sum_{i=0}^{\infty} \binom{i+k-1}{i} p^i s^{i+k}$$

Let now $i + k = n$

$$\phi(s) = \sum_{n=0}^{\infty} \left[e^{-\lambda} \sum_{k=1}^{n} \frac{1}{k!} \binom{n-1}{k-1} \lambda^k p^{n-k} \right] s^n$$

This shows the probability of a demand n to be

$$(3.4) \qquad f(n) = e^{-\lambda} \sum_{k=0}^{n-1} \binom{n-1}{n-k} \frac{1}{k!} \lambda^k p^{n-k}$$

The mean and variance of this distribution are

$$\mu_1 = \frac{\lambda}{1-p}$$

$$\mu_2 = \frac{\lambda(1+p)}{(1-p)^2}$$

These two moments of the distribution were fitted to the histogram of one flight with a sufficiently low load factor in order that all demands should fall within the capacity limits of the flights and a maximum number of observations became available.* A chi-square test did not reject the distribution at the 5 percent significance level. However an unexpectedly large value of p

*Fig. 4.

was obtained, $p = .50$ so that it remains doubtful that the distribution (4) really explains the observed frequencies. This p-value would predict the following probabilities of demand clusters:

1	seat	in	50 percent	of	all	cases
2	seats	"	25	"	"	"	"
3	"	"	12.5	"	"	"	"
4	"	"	6.25	"	"	"	"
more than 4	"	"	6.25	"	"	"	"

3.2.3. The Gamma distribution

The observed frequency distribution of manifested reservations tend to be more spread out than Poisson distributions are. One suspects that this is due not only to demand clustering but also to variations in the mean demand which occur in response to such random events as the weather, holidays, or events like conventions. This would mean that the Poisson parameter λ should itself be regarded a random variable. Among the simple unimodel distributions for this parameter that have some plausibility and considerable flexibility are the Gamma-distribution. [cf. (3.7)] It has been shown [cf. Kendall, pp. 124, 125] that if the Poisson parameter is Gamma distributed then the variable n obeys a negative binomial distribution

$$(3.5) \qquad f(n) = (1 - c)^m \binom{n+m-1}{n} c^n$$

where c is a constant ≥ 1 and m a positive integer. The question arises whether this somewhat complex distribution may be modified for group demand and whether in this way a more elementary distribution may be obtained.

The generating function of (5) is $(\frac{c}{c+1})^m (1 - \frac{s}{c+1})^{-m}$. Substituting the generating function of (3) for s in (5) we obtain after some straightforward calculation.

$$(3.6) \qquad f(n) = (1 - c)^{m} \sum_{k=1}^{n} \binom{k-m-1}{m-1} \binom{n-1}{k-1} p^{n-k} (c - cp)^{k}$$

Now
$$\sum_{k=1}^{n} \binom{n-1}{k-1} p^{n-k} (c - cp)^{k-1} = (p + c - cp)^{n-1}$$

The expression (6) may therefore be written

$$f(n) = C\binom{\theta n+m-1}{m-1} \cdot d^{n}$$

where C,d are positive constant and θ, 0 < θ < 1 is some constant. In other words (3.6) represents a weighted average of the terms $\binom{k-m-1}{k}$, multiplied by the sum of the weights. If θ remains reasonably constant as n is varied and if m is not too large the expression $\binom{\theta n+m-1}{m-1}$ is approximately proportional to a (m-1)st power of n. Since the assumptions about the distributions of demand by groups and the distribution of Poisson parameter are approximations, too, we need not be meticulous. The upshot is that

$$f(n) \approx c_{o} \cdot n^{m} \cdot c^{n}$$

which is the discrete variant of the so-called Gamma distribution

$$(3.7) \qquad f(x)\, dx = \frac{\alpha^{k} x^{k-1}}{\Gamma(k)} e^{\alpha x} dx$$

For the cumulative distribution we shall use the notation

$$(3.8) \qquad \int_0^x \frac{\alpha^k u^{k-1}}{\Gamma(k)} e^{-\alpha u} du = \int_0^{\alpha x} \frac{u^{k-1} e^{\alpha u}}{\Gamma(k)} du = \Gamma_{\alpha x}(k).$$

The mean and variance of the Gamma distribution are respectively,

$$\mu_1 = \frac{k}{\alpha}$$

$$\mu_2 = \frac{k}{\alpha^2} .$$

The mean of the hypothetical Gamma distribution was fitted to the observed means and then a maximum likelihood estimate of the exponent k was obtained for two flights. (fig. 4, 5, 6)

In view of the fact that no observations were left out for the entire 10 month period which included the winter season the agreement is good enough to accept the Gamma distribution. More work would be needed to establish its universal applicability. The exponent k is identical with that which occurs in the Gamma distribution of the Poisson parameter. We should expect this exponent to vary little from one flight to another, since the manner in which the mean demand fluctuates should be fairly similar for all flights. This is borne out since $\hat{k}$ varies between 3.7 and 4.9 for the flights observed.

3.3. Losses

3.3.1. No shows.

As the simplest hypothesis one might assume that the probability of anyone being a no show is constant. Then the distribution of no shows, given the number of passengers manifested, is binomial with a parameter given by the number of passengers manifested.

The no show distribution is of interest only when the number of passengers manifested equals or exceeds capacity. Then, the exponent is large and the probability of any particular passenger being a no show is small, whereas the product of the two is of moderate size -- about $\frac{1\frac{1}{2}}{10}$. Under these conditions the binomial distribution may be approximated by a Poisson distribution. With the refinements discussed in the case of the demand for reservations, the Poisson distribution turns into a Gamma distribution.

Observation shows that occasionally there is still another effect at work causing zero no shows to have a higher probability. This is perhaps due to the fact that some no shows occur only under unfavorable conditions which are present only with some probability.

The number of observations yielding no show distributions was a fraction of the observations for manifested reservations, because in order to have an observation, manifested passengers had to be in the neighborhood of capacity. Moreover pertinent data had not been recorded for more than 7 months. For this reason the empirical evidence is insufficient to reveal the nature of the no show distribution in any great detail. The impression is given by the histograms, however, that the no show distribution may fall into the orbit of Gamma distributions. (fig. 7 Nos. 4, 5, 6).

3.3.2. Late Cancellations and Misconnections

The histogram shows that the frequency of cancellations is a rapidly decreasing function of their number. The underlying tendency appears to be a Poisson distributed "rare event" on which is superimposed the effect of clustering and perhaps of variations in the parameter. The outcome is presumably again a Gamma distribution. (fig. 7, No. 1)

The same may be said about misconnections. (fig. 7, Nos. 2, 3)

3.3.3. Total Losses

Suppose that the three elements: no shows, late cancellations and misconnections are each subject to a Gamma distribution. A sum of Gamma distributed variables is itself Gamma distributed only if the coefficients α (in formula 3.7) agree. This is not likely the case here. However late cancellations and misconnections are usually small compared with no shows. The distribution of total losses may thus be regarded as a modified distribution of no shows.

Since it is the total losses that matter in the decisions discussed later (the sale goal and the stop sales problem) their distribution has been analyzed more carefully. (fig. 8, 9). The approximation by Gamma distributions is satisfactory, and the exponents are agreeably close to each other and to those of the demand distributions.

3.4. Stand By Passengers

The population of stand by passengers consists of through passengers who missed connections or changed plans, and of local passengers who either did not bother about making a reservation or who took a chance upon being unable to secure a reservation in time. The superposition of these demand elements may produce complex shaped distributions. Histograms show the comforting fact that except for an occasional underrating of zero demands, Gamma distributions will fit as well as anything.

The observations cover a period of only two months and only such flights for which demand was low enough to permit the total demand of stand bys to be accommodated. These cases disregard the possibility of a demand in the category "unable to secure a reservation". However, on ordinary days the demand in this category is slight so that local stand bys are mostly of the "non reserving" kind.

Another class of additional passengers are "errors", i.e. non-manifested passengers with reservations. Under the present system they are said to be not numerous enough to be of effect in any of the following calculations. No observations were available.

4. Choice of Sales Limits

4.1. By "sales limit" we denote the maximal number of reservations that it is optimal to sell on a given flight. In general not all the passengers with valid reservations and tickets will show up or cancel their reservations in good time. This fraction is on the average about 10 percent. Since the passenger is entitled to a full refund for his ticket, -- except in tourist flights where a 20 percent deduction or a minimum deduction of 5 dollars is made -- the airline loses the revenue of his seat. If the airline sells reservations in excess of capacity it risks being caught short. The ensuing loss in good will per seat may amount to a multiple of the revenue.

What is the best policy in this dilemma? Some compromise between both types of uncertain losses has to be found. The practice seems to be that the average frequency of excess passengers relative to the number of flights is subjected to a fixed limit. This limit is in the neighborhood of 1 percent.

But how can we justify the particular limit that is being chosen? For a rational choice it is inevitable that a definite money value is assigned to the average loss of goodwill that results from retaining or inconveniencing a passenger -- which essentially is a loss of future sales to this passenger and individuals persuaded by him. Some of the perplexing features of the load planning problem arise from the fact, that the largest over and under sales tend to occur in connection with special days such as holiday weekends. The factors influencing the demand for transportation and the public's propensity for not keeping reservations are so various and unpredictable that no formal analysis can measure up to the experience of reservation managers. In our analysis which proceeds on the assumption that probability distributions are constant and known, we have to concentrate on the routine days and decisions, leaving aside the special day problem.

From the description of the reservation process in the preceding section it appears that the reservations policy must fix two parameters: the desired reservations level -- the sales goal -- and the "cushion". The function of the latter is auxiliary: to prevent overshooting the mark of the sale limit while at the same time bringing sell and record reservations as near to it as feasible.

In a complete treatment of the reservations problem the following would be the object of the policy: to maximize the sum of revenue minus penalties for removals minus costs of reservations for every flight subject to its capacity limitation. Notice that operating cost may be treated as fixed since the schedules and capacities of all flights are given in the problem.

In this generality the problem is too complex for practical analysis. We propose to isolate the two essentially different subproblems that are involved: the determination of the sale limit and of the sales cushion. The first problem is a balancing of two sources of loss, each of a large order of magnitude relative to reservations cost proper -- which will therefore be disregarded. These costs are lost revenue from unused seats and penalties for inconveniencing or returning passengers. The second problem involves primarily an equilibration of communications cost. It will be considered in section 5.

<u>4.2. Analysis</u>

<u>4.2.1. Assumptions</u>

4.2.1.1. The probability distribution of losses (late cancellations, no shows, and misconnections) is conditional on the number of reservations manifested.

But in the neighborhood of the capacity of the flight, the distribution is not very sensitive to the exact number of manifests so that the latter will be disregarded as a conditional variable later on.

4.2.1.2. Similarly the demand by stand by passengers is not independent of the number of passengers manifested because some stand bys may have been unable to secure a reservation. But in practice this effect is small, on normal days at least, and will be disregarded here.

4.2.1.3. The same average penalty will be attached to accommodating passengers in the airplane lounge and to returning him at the gate, because the first should be regarded as just a lucky escape for the airline.

4.2.1.4. "Errors" will be disregarded so that "adds" become identical with "stand by" passengers.

4.2.2. Notation

n	the number of reservations demanded until manifesting time
N	the sales limit
r	the number of losses
k	the number of "adds"
f(n)	the probability of a demand for n reservations
p(r,n) or p(r)	the probability of r losses
q(k)	the probability of a demand by k stand bys.
a	the loss per unutilized seat
b	the penalty per excess passenger
c	the capacity of the flight.

4.2.3. Statement of the Problem

1) If $n - r > c$ there are $n - r - c$ excess passengers and the penalty is $b.(n - r - c)$

2) If $n - r \leq c$ and $n - r + k < c$ there is a deficit of $c - n + r - k$ passengers and the revenue lost is $a.(c - n + r - k)$

3) If $n - r \leq c$ and $n - r + k \geq c$ no loss or penalty is incurred.

For given $n \leq N$ the expected penalty in case 1) is

$$(4.1) \qquad \sum_{r=0}^{n-c} b.(n - r - c)\, p(r,n)$$

If $n > N$ the n in (1) is replaced by N. The probability of $n > N$ is $1 - F(n)$. Thus the expected penalty letting n vary is

$$(4.2) \qquad b \sum_{n=c}^{N} \sum_{r=0}^{n-c} (n - r - c)\, f(n)\, p(r,n) \;+ b\,[1 - F(N)] \sum_{r=0}^{N-c} (N-r-c)\, p(r,N)$$

In case 2) given $n \leq N$, the expected loss of revenue is

$$a \sum_{r=n-c}^{n} \sum_{k=0}^{c+r-n} (c - n + r - k)\, p(r,n)\, q(k)$$

Taking the expectation over n as in case 1) gives

$$(4.3) \qquad a \sum_{n=0}^{N} \sum_{r=n-c}^{n} \sum_{k=0}^{c+r-n} (c + r - n - k)\, f(n)\, p(r,n)\, q(k)$$

$$+\, a \sum_{r=N-c}^{N} \sum_{k=0}^{c+r-N} (c + r - N - k)\, [1 - F(N)]\, p(r,N)\, q(k)$$

The sum of (2) and (3) is to be minimized with respect to N. In the formulation we have tacitly assumed that $N \geq c$: Obviously it would not be profitable to refuse reservations before the capacity level is reached.

4.2.4. Our analysis of demand has shown that the distributions of reservations, losses and adds available may be approximated by continuous Gamma distributions. To facilitate calculations we shall now replace the discrete variables n, r, k by continuous variables x, u, v and the sums by integrals. The minimand is then

$$(4.4) \qquad b \int_c^N dx \int_0^{x-c} du \,(x-u-c)\, f(x)\, p(u,x) + b\left(1 - \int_0^N f(x)\,dx\right) \cdot \int_0^{N-c} (N-c-u)\, p(u,N)\, du$$

$$+ \; a \int_0^N dx \int_{x-c}^{x} du \int_0^{c+u-x} dv\,(c+u-x-v)\, f(x)\, p(u,x)$$

$$+ \; a\left(1 - \int_0^N f(x)\,dx\right) \int_{N-c}^{N} du \int_0^{c+u-N} dv\,(c+u-N-v)\, p(u,N)\, q(v)$$

Differentiation with respect to N yields

$$b\left(1 - \int_0^N f(x)\,dx\right) \cdot \int_0^{N-c} p(u,N)\,du + a\left(1 - \int_0^N f(x)\,dx\right)\Bigg[\int_0^c (c-v)$$

$$\cdot\; p(N,N)\, q(v)\, dv - \int_{N-c}^{N} du \int_0^{c+u-N} dv\, p(u,N)\, q(v)$$

$$+ \int_{N-c}^{N} du \int^{c+u-N} dv\,(c+u-N-v)\, \frac{\partial p(u,N)}{\partial N}\, q(v)\Bigg]$$

A necessary condition for N to yield a minimum is that this expression be zero. The term $1 - \int_0^N f(x)\,dx$ thus drops out and the distribution of the demand for reservations is seen to be irrelevant for the optimal N. If the effect of N on the conditional distribution of losses u is disregarded we obtain the simpler formula:

$$(4.5) \qquad \frac{b}{a} \int_0^{N-c} p(u)\,du + p(N) \int_0^c (c-v)\,q(v)\,dv - \int_{N-c}^{N} p(u) \int_0^{c-N+u} q(v)\,dv\,du = 0$$

Introducing excess sales $N - c = y$ as a new variable this equation is further reduced to

$$\frac{b}{a} \int_0^y p(u)\,du + p\,(c + y) \int_0^c (c - v)\,q\,(v)\,dv = \int_0^c p(u + y) \int_0^u q(v)\,dv\,du$$

In particular if p and q are Gamma distributions with integral exponents

$$p(u) = \frac{\beta^j u^{j-1}\,e^{-\beta u}}{\Gamma(j)}$$

$$q(v) = \frac{\gamma^i v^{i-1}\,e^{-\gamma v}}{\Gamma(i)}$$

$$(4.6) \qquad \frac{a+b}{a}\ {}_{\beta y}(j) + \frac{\beta^j (c + y)^{j-1}}{(j - 1)!}\,e^{-\beta(c+y)} \cdot \left[c\,\Gamma_{\gamma c}(i) - \frac{i}{\gamma}\,\Gamma_{\gamma c}(i+1)\right]$$

$$= -\sum_{r=0}^{j-1} h_r\,y^r\,e^{-\beta y} + \Gamma_{\beta(c + y)}(j)$$

where

$$h_r = \frac{\beta^j}{r!} \sum_{k=0}^{i-1} \frac{(k + j - r - 1)!}{(\beta + \gamma)^{k+j-r}} \Gamma_{c(\beta+\gamma)}(k+j-r) \tag{4.7}$$

is independent of y. Or writing $H = c \Gamma_{\gamma c}(i) - \frac{1}{\gamma} \Gamma_{\gamma c}(i + 1)$

$$\sum_{r=0}^{j-1} h_r y^r e^{-\beta y} + (1 + \frac{b}{a}) \Gamma_{\beta y}(j) = \Gamma_{\beta(c+y)}(j) - H \frac{(c+y)^{j-1} \beta^j e^{-\beta(c+y)}}{(j - 1)!} \tag{4.8}$$

where the left hand members are functions of y and the right hand members are functions of $c + y$. The last term on the right hand side is usually quite negligible. Since the solution of the equation must be an integer and its order of magnitude is known, trial and error leads to rapid convergence.

<u>Examples:</u>

Space available	c	=	58
average loss (turnover)	$\frac{j}{\beta}$	=	5
exponent of loss distribution	j	=	4
average stand bys	$\frac{i}{\gamma}$	=	3
exponent of stand by distribution	i	=	4
average load factor	$\frac{k}{\lambda c}$	=	.86

ratio of penalty to revenue per seat

$\frac{b}{a}$ = 100 75 50 25 10 5 2

optimal sales limit

N = 59 59 59 59 60 60 61

5. Various Calculations.

5.1. The Cushion Problem

Notation

m	sales during the lag period
$d(m)$	distribution of sales during the lag period
s	the stop sales level
$x = N - s$	the cushion
g	extra communications cost plus probability of a lost sale for every unsold seat of the cushion
h	extra communication cost plus risk of capacity shortage for every seat sold beyond the sales limit
μ	the probability of receiving a reservation request (for an unspecified number of seats) during the lag period

1) If $s + m > N$ then there are $(N - s - m)$ excess reservations carrying a cost of $h \cdot (s + m - N)$

2) If $s + m < N$, $n - s - m$ or $N - s - m$ calls for reservations, whichever is smaller, must be handled at a communications cost $g \cdot (n - s - m)$ or $c \cdot (N - s - m)$.

The minimand is

$$h \sum_{m=N-s+1} (s + m - N)\, d(m) + g \sum_{n=s}^{N} \sum_{m=0}^{n-s} (n - s - m)\, f(n)\, d(m)$$

$$+ \; g \sum_{n=N+1}^{\infty} \sum_{m=0}^{N-s} (N - s - m)\, f(n)\, d(m)$$

A necessary condition for a minimum is that the first difference with respect to s should be approximately zero.

$$0 \approx h \sum_{m=N-s+1}^{\infty} d(m) - c \sum_{h=s+1}^{N} f(n) \sum_{m=0}^{n-s-1} d(m)$$

$$- c \sum_{n=N+1}^{\infty} f(n) \cdot \sum_{m=0}^{N-s-1} d(m)$$

Replacing s by the cushion $x = N - s$ we have

$$(5.1) \qquad h \sum_{m=x+1}^{\infty} d(m) = c \sum_{n=N-x+1}^{\infty} f(n) \sum_{m=0}^{n-(N-x)-1} d(m) + c \sum_{n=N+1}^{\infty} f(n) \cdot \sum_{m=0}^{x-1} d(m)$$

Let now the distribution of the probabilities of sales during the long period be the one given in section 3

$$(5.2) \qquad d(m) = \begin{cases} 1 - \mu & m = 0 \\ \mu\,(1 - p)\,p^{m-1} & m > 0 \end{cases}$$

Then (5.1) simplifies to

$$(5.3) \qquad p^{-x} \sum_{n=N-x+1}^{\infty} f(n) - \mu \sum_{n=N-x+1}^{\infty} p^{n-N-1} f(n) = \frac{h\mu}{c} + \frac{\mu}{p} \sum_{n=N+1}^{\infty} f(n)$$

where all terms containing x are on the left hand side.

We now consider the demand distribution $f(n) = \dfrac{n^{k-1}\alpha^k e^{-\alpha n}}{(k-1)!}$

Writing

$$\Gamma_x(k) = \int_0^x \frac{u^{k-1}}{\Gamma(k)} e^{-u}\, du \qquad \text{for the incomplete Gamma}$$

function and replacing sums by integrals, formula (5.3) becomes

$$(5.4)\qquad p^{-x}\left[1-\Gamma_{\alpha(N-x+1)}(k)\right]-\frac{\mu p^{-(N+1)}}{(1-\frac{\log p}{\alpha})^{k}}\Gamma_{(\alpha-\log p)(N-x+1)}(k)$$

$$=\frac{h\mu}{g}+\frac{\mu}{p}\Gamma_{\alpha(N+1)}(k)$$

For parameter values that occur in our observations the complicated second term on the left hand side is small enough to be dropped.

Example*

Average total demand for reservations	αk	=	40
average demand during the lag period	$\frac{\mu p}{1-p^2}$	=	1
parameter in the distribution of the number of seats demanded in a reservation request	p	=	.7
exponent of the demand distribution	k	=	5
ratio of costs	$\frac{h}{g}$	=	2:1
cushion	x	=	5

This means a stop sales level of 55. If the cost ratio is $\frac{h}{g} = 5:1$ instead the stop sales level is at 53. This "cushion" is large compared with the size of those used in practice. It suggests that the cost of stopping sales before full capacity is regarded as higher relative to the cost of "protecting sales" than was assumed in this example.

5.2. Average Total Demand and the Load Factor

The total demand for a flight will be defined to be the number of reservations requested minus losses plus errors and stand bys. The exact distribution of the total demand, given that reservation demand, losses, and stand bys are Gamma distributed is not easy to calculate but will be approximately a Gamma

* Mr. R. Muth has kindly undertaken to calculate this example.

distribution, if losses and stand bys are a relatively small fraction of reservations. Empirical observations confirm that the distribution of total passengers carried is approximately a Gamma distribution. (fig. 11). We shall assume that the exponent of the Gamma distribution of total demand is the same as that of the distribution of reservations.

The mean number of passengers carried is

$$M = \int_0^c x^k \frac{\lambda^k e^{-\lambda x}}{\Gamma(k)} dx + c \int_c^{\infty} \frac{x^{k-1}\lambda^k e^{-\lambda x}}{\Gamma(k)} dx$$

$$(5.5) \qquad = \frac{k}{\lambda} \Gamma_{\lambda c}(k+1) + c \left[1 - \Gamma_{\lambda c}(k) \right]$$

This compares with a mean total demand of $\frac{k}{\lambda}$. The ratio of the mean number of passengers carried to the capacity c is known as the average <u>load factor</u>

$$p = \frac{k}{\lambda c} \Gamma_{\lambda c}(k+1) + 1 - \Gamma_{\lambda c}(k)$$

Introducing the mean demand $m = \frac{k}{\lambda}$

$$p = 1 + \frac{m}{c} \Gamma_{\frac{kc}{m}}(k+1) - \Gamma_{\frac{kc}{m}}(k)$$

For instance assuming $k = 5$ a mean demand of 50 passengers on a flight of capacity 58 will induce a load factor of $1 + \frac{50}{58} \Gamma_{5.8}(6) - \Gamma_{5.8}(5) \approx 0.77$.

5.3. Average Rate of Excess Passengers

Using the notation of section 4.2.3 the probability of having j passengers in excess of capacity is $\sum_{r=0}^{N-(c+j)} f(r + c + j)\, p(r, r + c + j)$

The average number of excess passengers per flight therefore is

$$E = \sum_{j=1}^{N-c} j \sum_{r=0}^{N-(c+j)} f(r + c + j)\, p\,(r, r + c + j)$$

Let f and p be again Gamma distributions. As a further simplification we approximate $f(r + c + j)$ by $f(c + \theta y)$ for all j $0 \leq j \leq N-c$ where θ is some fixed parameter, $0 \leq \theta \leq 1$. Thus

$$E = f(c + \theta y) \int_0^y z \int_0^{y-z} p(x)\, dr\, dz$$

$$= \frac{(c+\theta y)^{k-1} \alpha^k}{\Gamma(k)} e^{-\alpha(c+\theta y)} \cdot \int_0^y (y - z) \int_z^y \frac{\beta^j u^{j-1}}{\Gamma(j)} e^{-\beta u}\, du\, dz$$

$$(5.6) \qquad E = \frac{(c+\theta y)^{k-1} \alpha^k}{\Gamma(k)} e^{-\alpha(c+\theta y)} \left[y \frac{j}{\beta} \Gamma_{\beta y}(j+1) - \frac{1}{2} \frac{j(j+1)}{\beta^2} \Gamma_{\beta y}(j+2) \right]$$

This expression denotes the average ratio of excess passengers to flights. We may also ask the reverse question: given a ratio of excess passengers to flights of 1 percent, what is the sales limit N that is associated with it? To answer this E on the left hand side of (5.6) is set 0.01 and the equation is solved with respect to y.

Example

N	=	60		sales limit
c	=	58		capacity
y	=	2		difference
θ	=	$\frac{1}{2}$		fraction factor
$\frac{k}{\alpha}$	=	52	$k = 5$	mean reservation demand
$\frac{j}{\beta}$	=	5	$j = 4$	mean losses
E	=		0.0104	mean excess passengers

Conversely a rate of excess passengers of 1 percent may be regarded as due to a policy of overselling by 2 reservations.

Finally we may find the penalty per excess passengers that would have given rise to the same rate of excess passengers to flights. From (4.8) we have

$$(5.7) \quad \frac{b}{a} = \frac{\Gamma_{\beta(c+y)}(j)}{\Gamma_{\beta y}(j)} - \frac{N(c+y)^{j-1}\beta^{j} e^{-\beta(c+y)}}{\Gamma_{\beta y}(j)\,(j-1)!} - \sum_{r=0}^{j-1} \frac{h_r}{\Gamma_{\beta y}(j)} y^r e^{-\beta y} - 1$$

Example: For the data of the last example the table in the example of Section 4 tells us that an oversale of $y = 2$, giving $N = 60$, is associated with a penalty to revenue ratio of between 5:1 and 10:1. This then is the implication of a 1 percent oversale policy for what may be regarded a typical flight.

Flight X period 9/7 - 11/18, 1954 Tuesday through Friday only

dotted line - Poisson cumulative $\lambda = 46.3$
solid line - Observed cumulative

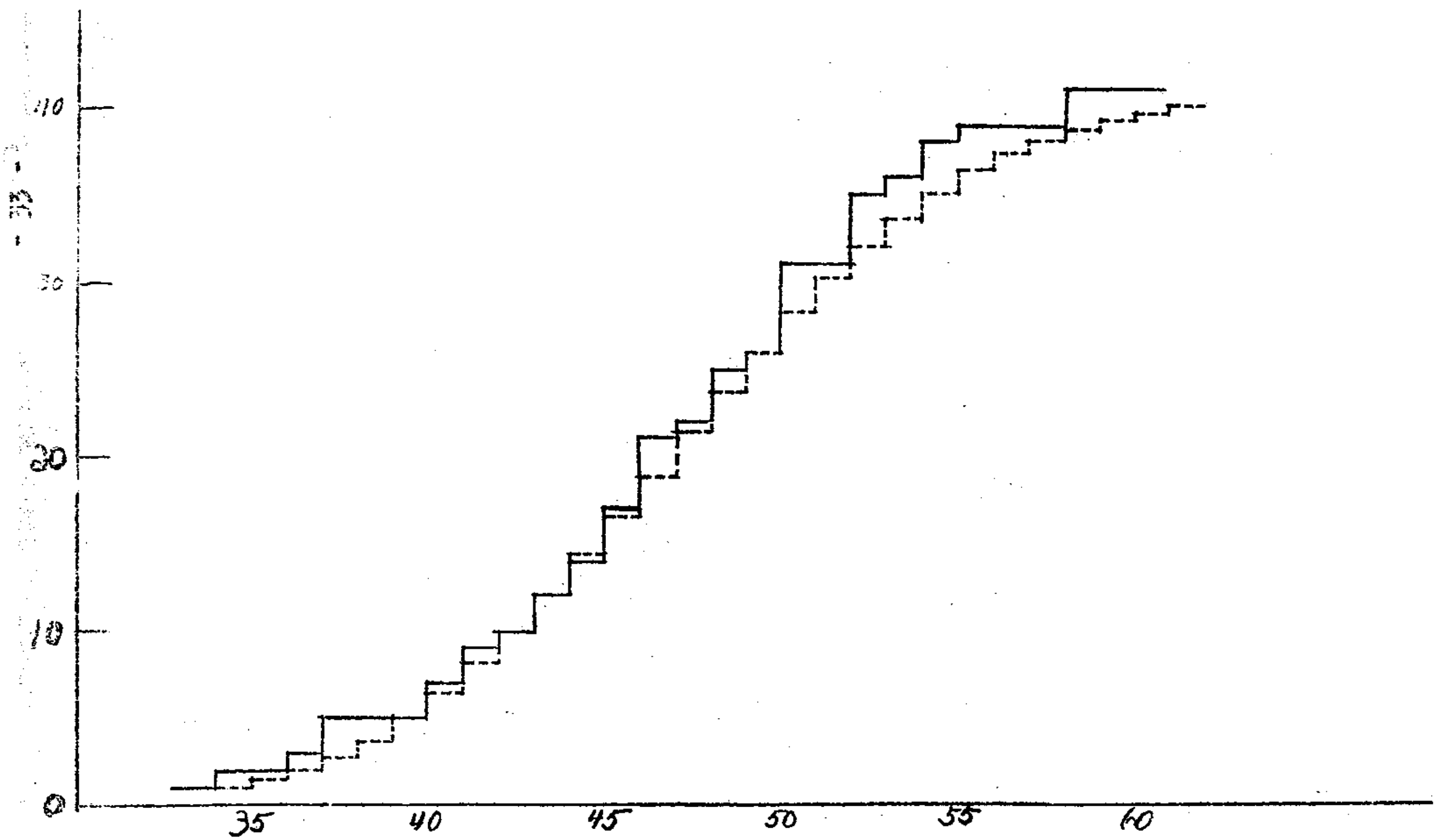

Fig. 1.

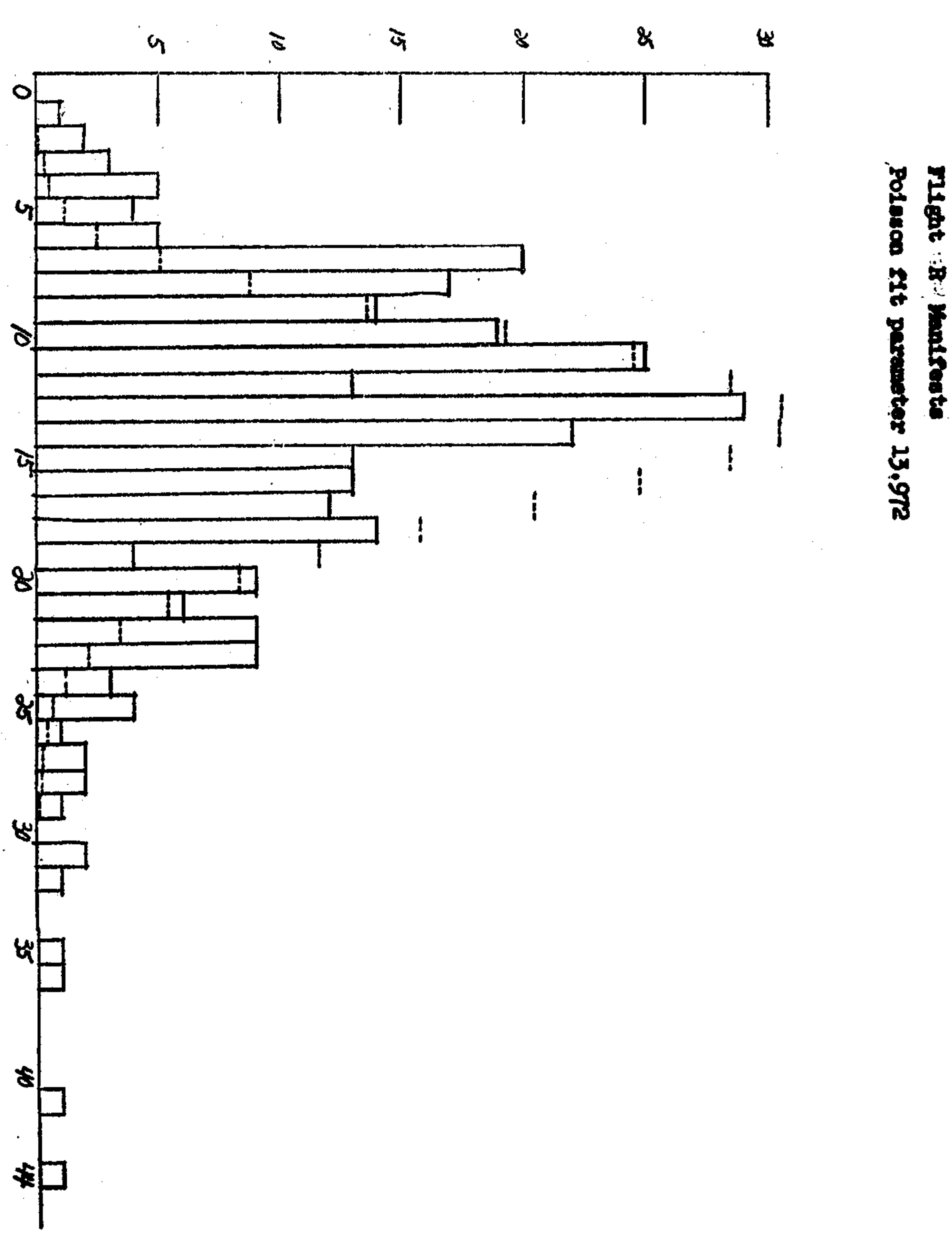
Flight "R" Manifests
Poisson fit parameter 13.972
5
10
15
20
25
30
0
5
10
15
20
25
30
35
40
44

Fig. 2.

Flight X Period 9/7 - 11/18, 1954 Tuesday through Friday only
dotted line - Poisson Frequency distribution λ = 463
solid line - observed frequencies.

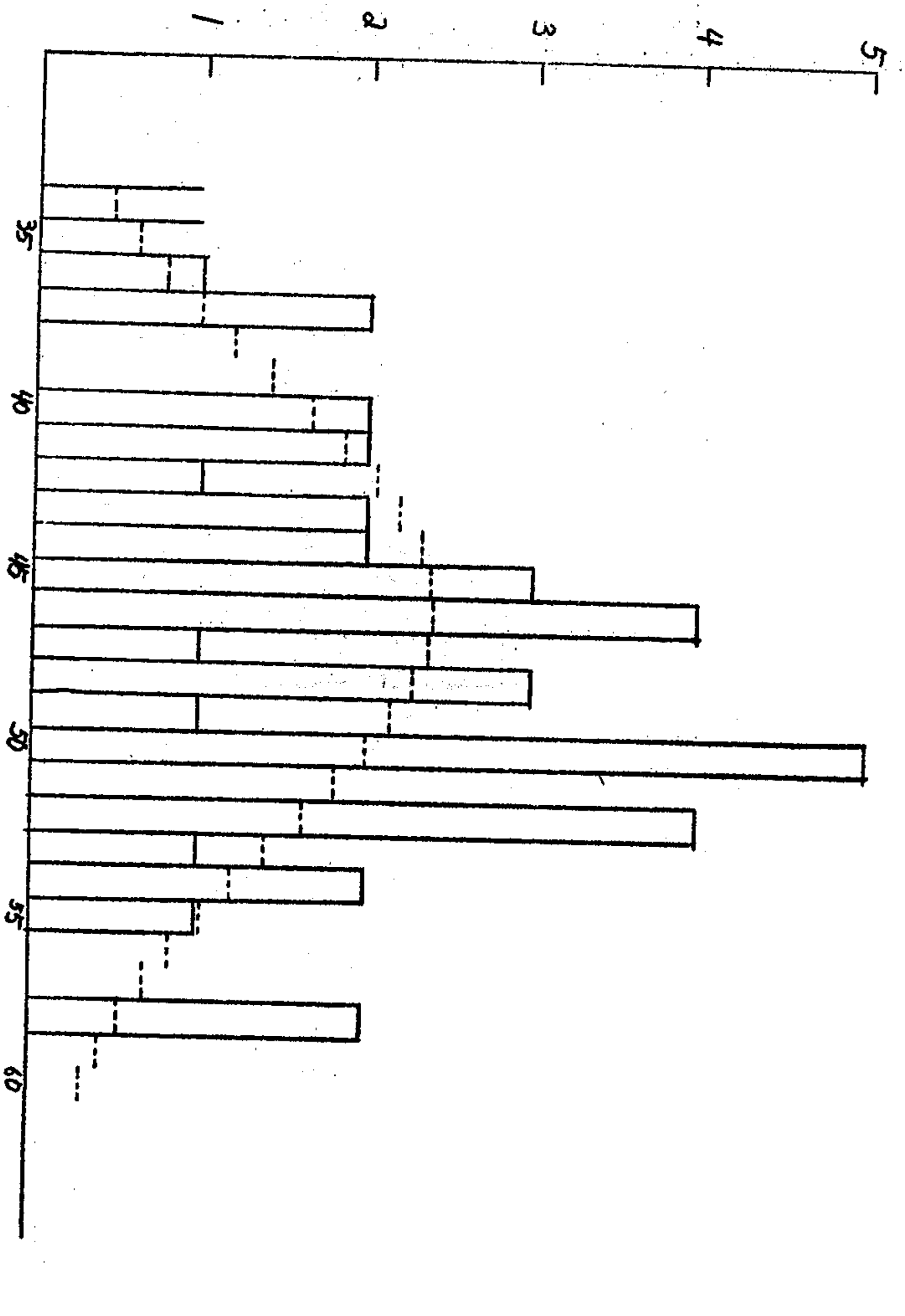

Fig. 3.

Flight Y manifests

Curved line -- Maximum Likelihood fit of (x + 1/2) to P density; $\frac{\alpha^{\beta}}{P(\beta)} x^{\beta-1} e^{-\alpha x}$ $\beta = 4.94991$ $\alpha = .148541$

Heavy short lines -- Method of moments fit to compound Poisson with Geometric

Parameter in Geometric .506
Parameter in Poisson 6.903

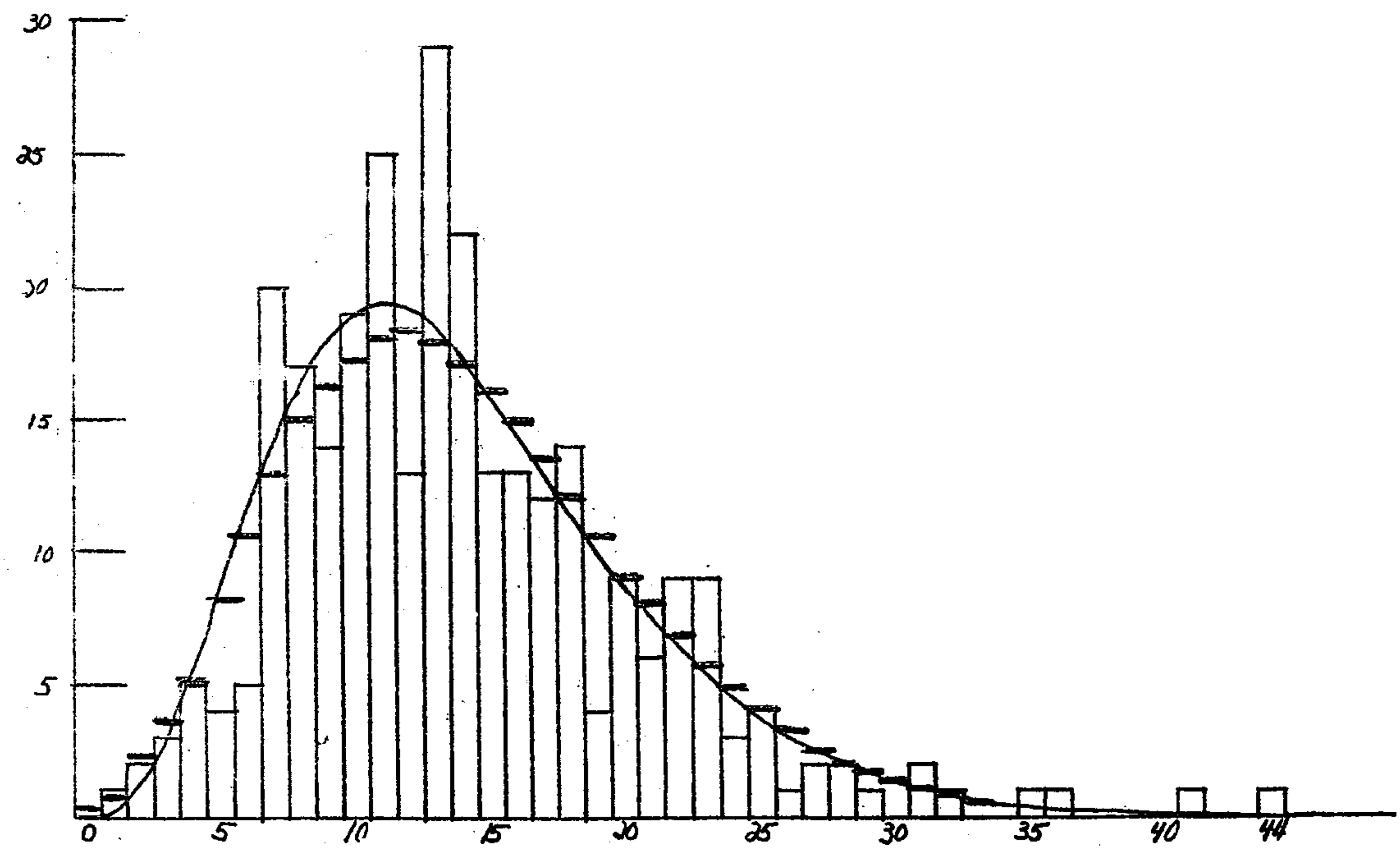

Fig. 4.

Flight 2

Fit of $\Gamma_x(\alpha,\beta) = \frac{\alpha^\beta}{P(k)} \int_0^x t^{k-1} e^{-\alpha t}\, dt$

dotted curve - maximum likelihood estimate - $\hat{\alpha} = .34$, $\hat{\beta} = 4.96$

solid curve - moments estimate - $\hat{\alpha} = .25$, $\hat{\beta} = 3.69$

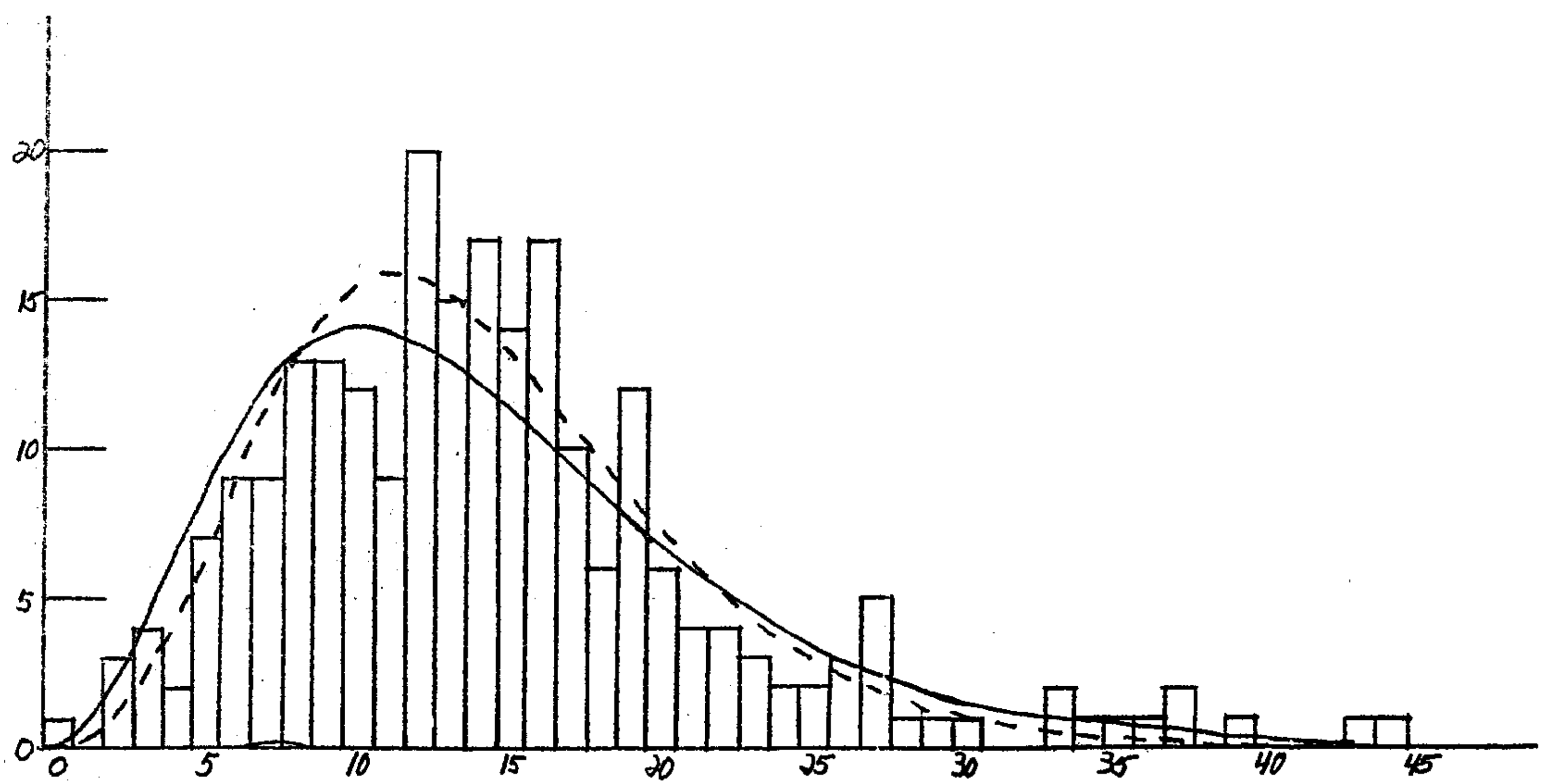

Fig. 5.

Flight x Manifests   Curved line -   fit to $x + 1/2$

$\hat{\alpha} = .251922$

$\hat{k} = 3.78044$

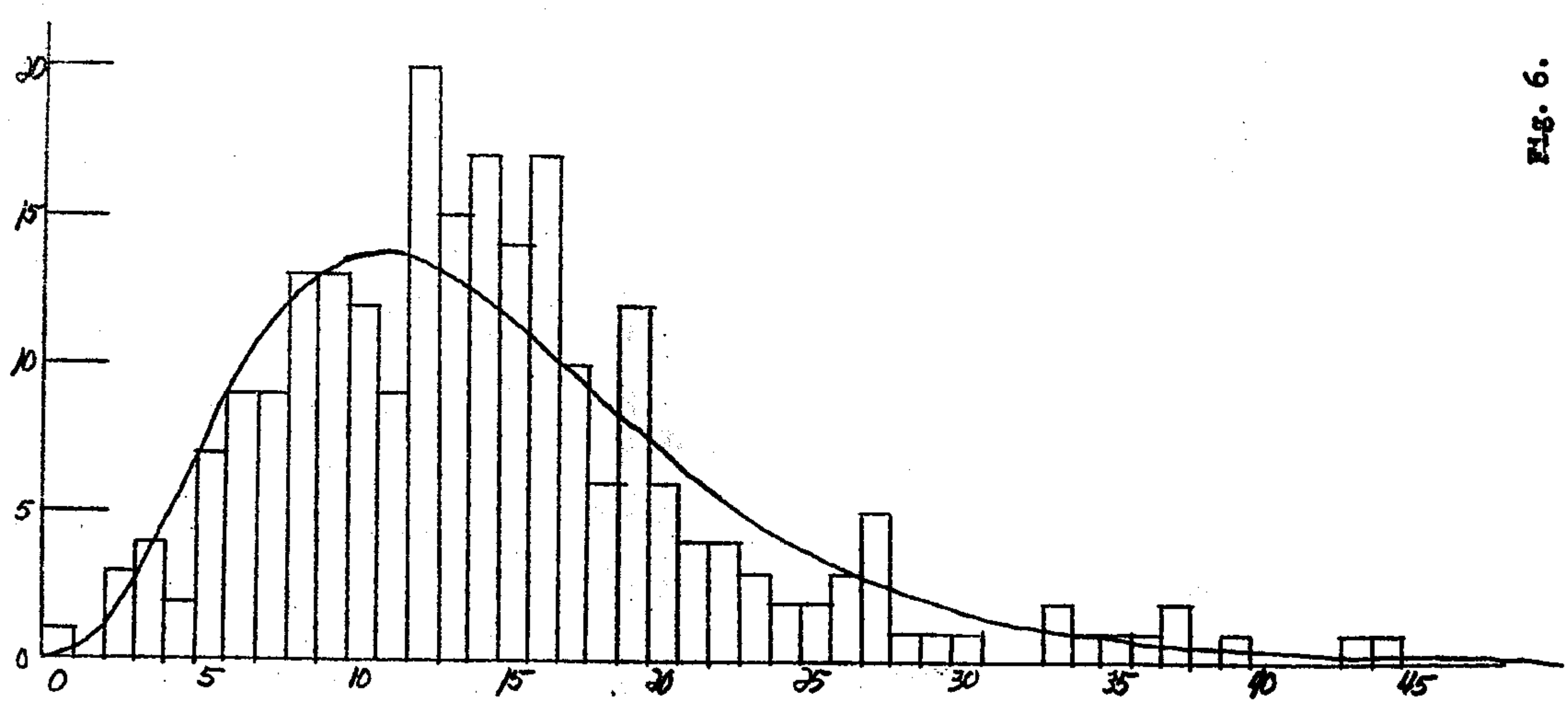

Fig. 6.

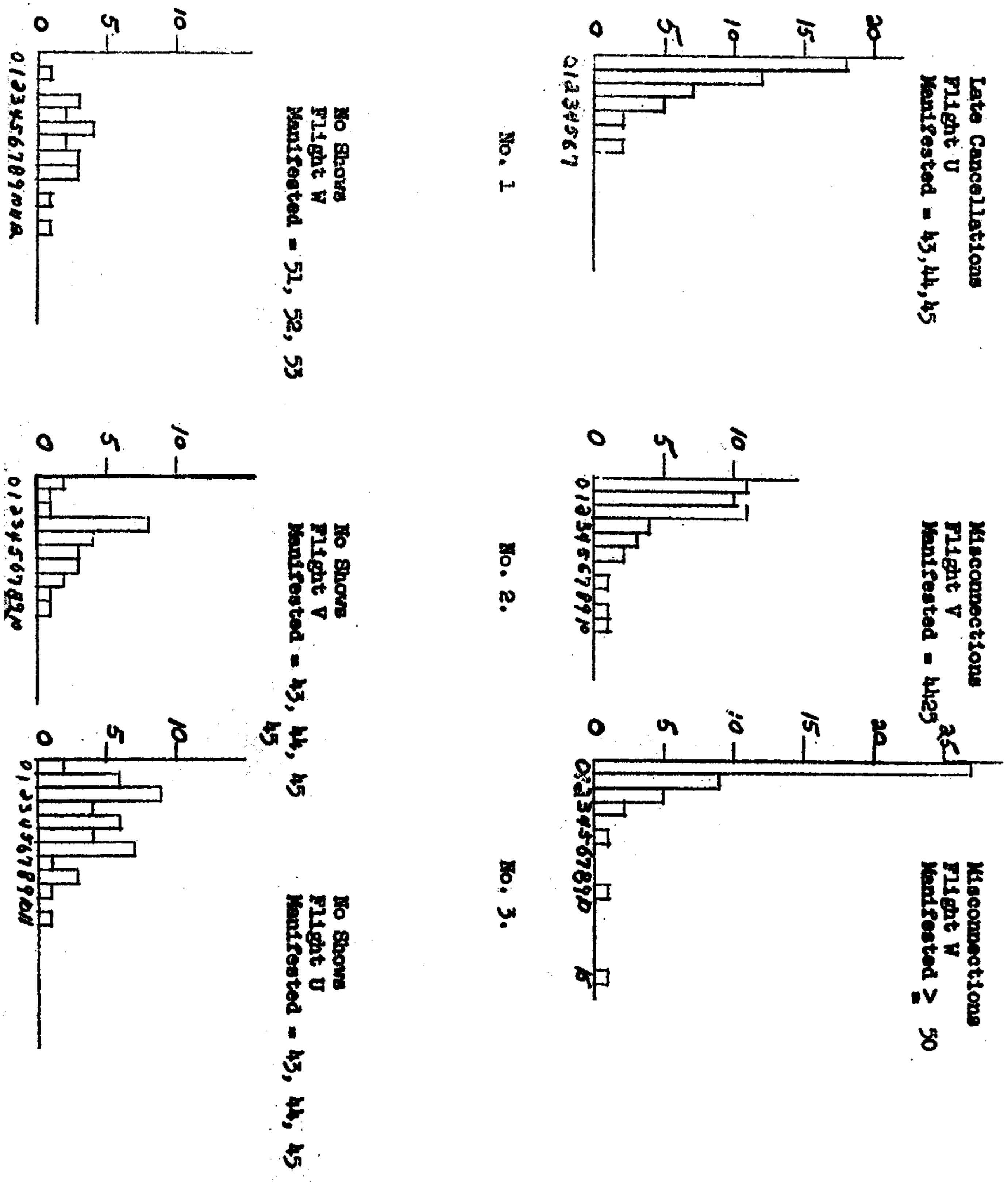
Late Cancellations
Flight U
Manifested = 43,44,45
No. 1
Misconnections
Flight V
Manifested = 44
No. 2.
Misconnections
Flight W
Manifested ≥ 50
No. 3.
No Shows
Flight W
Manifested = 51, 52, 53
No Shows
Flight V
Manifested = 43, 44, 45
No Shows
Flight U
Manifested = 43, 44, 45

Fig. 7.

Flight Q LC + NS + MS for manifests of 50 and over

Γ distribution using (x + 1/2) $\hat{\alpha}$ = .477582 $\hat{\beta}$ = 4.25723

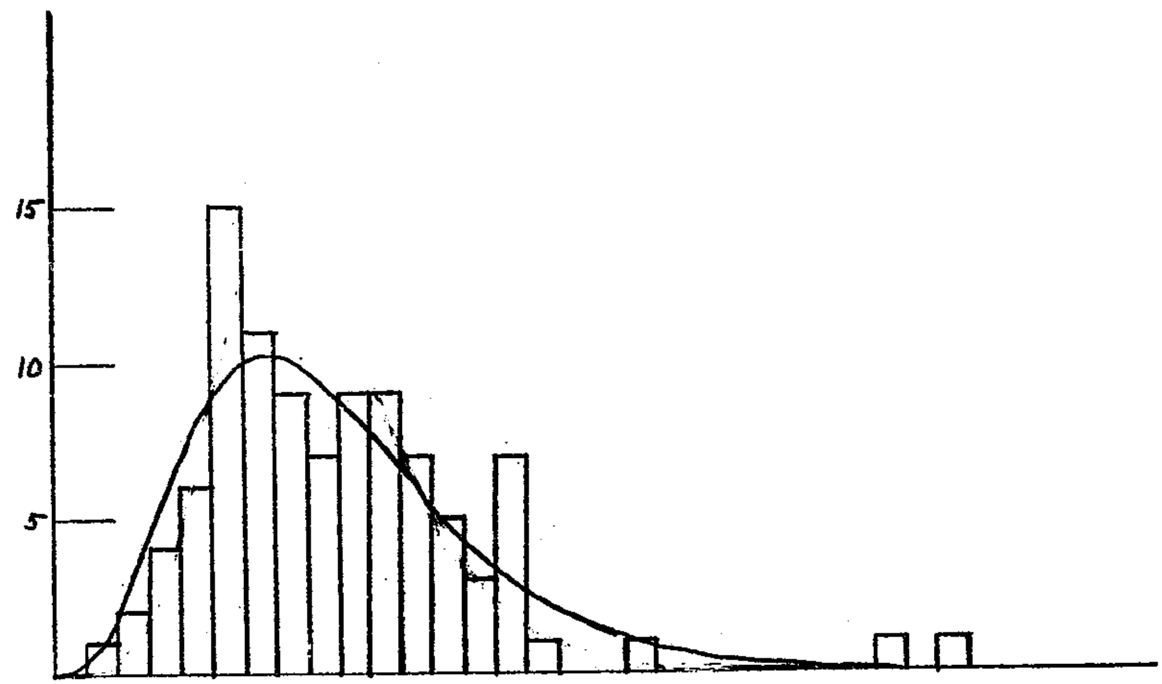

Fig. 8.

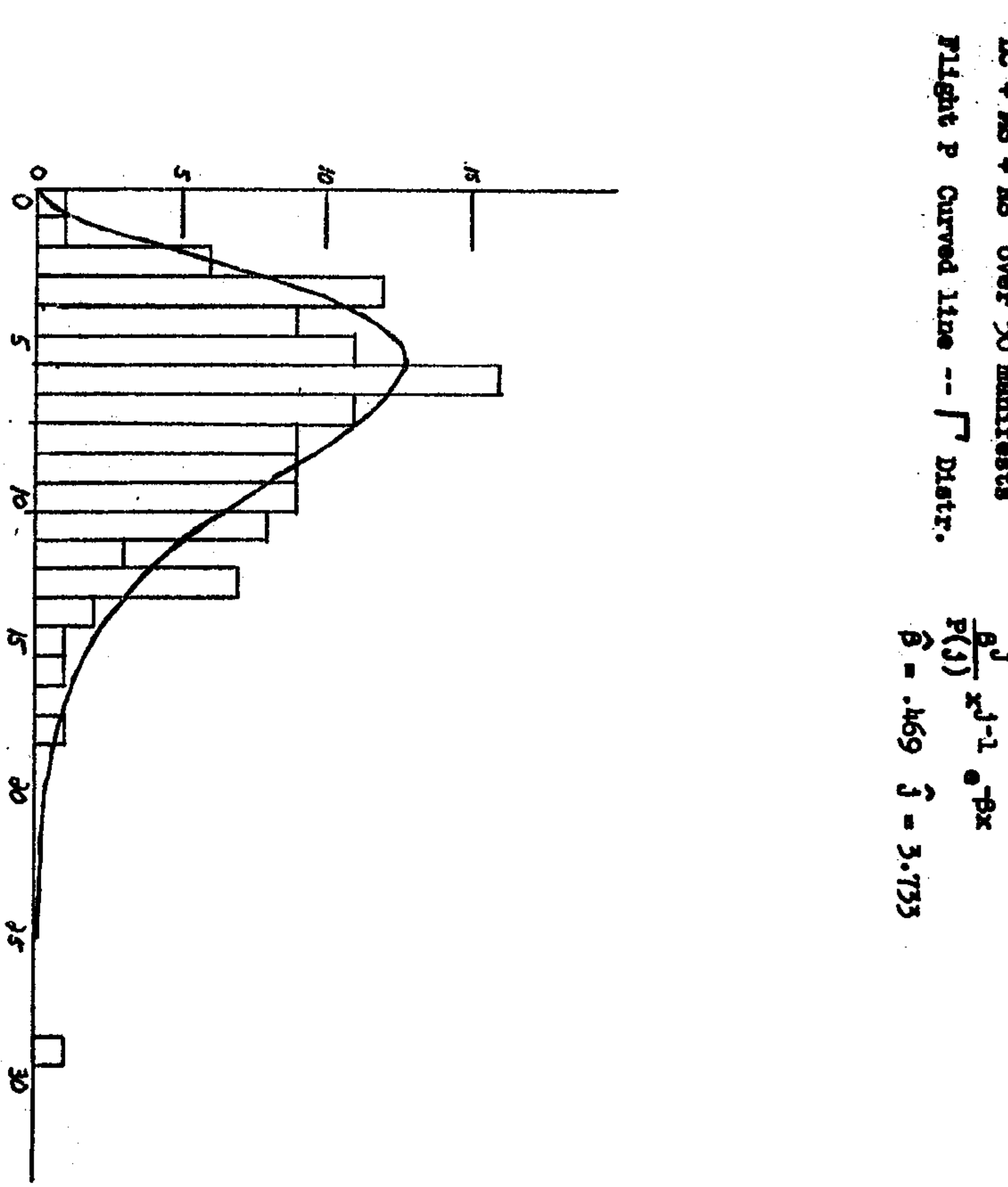
LC + MS + MS over 50 manifests
Flight P Curved line -- Γ Distr.
$\frac{\beta^j}{\Gamma(j)} x^{j-1} e^{-\beta x}$
$\hat{\beta} = .469 \quad \hat{j} = 3.733$
0
5
10
15
20
25
30

Fig. 9.

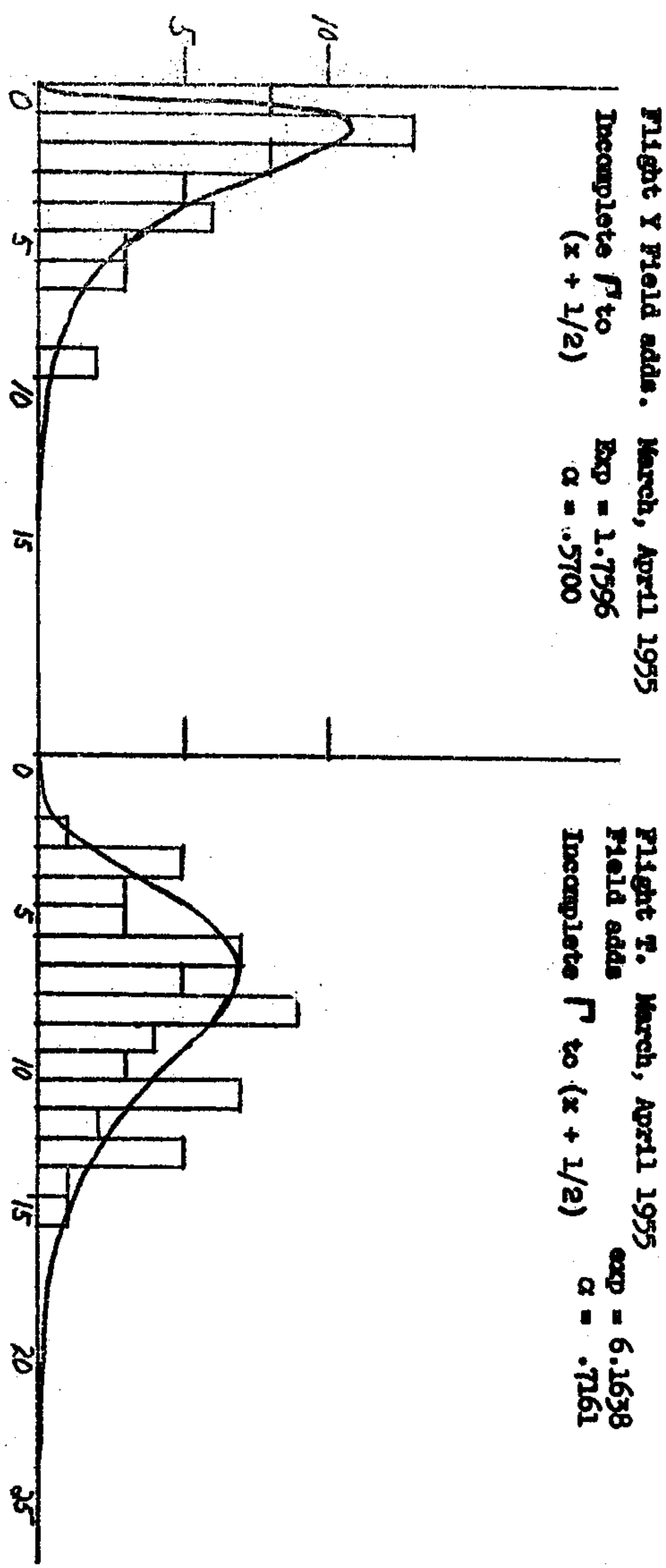
Flight Y Field adds. March, April 1955
Incomplete Γ to (x + 1/2)
Exp = 1.7596
α = .5700
Flight T. March, April 1955
Field adds
Incomplete Γ to (x + 1/2)
exp = 6.1638
α = .7161

Fig. 10.

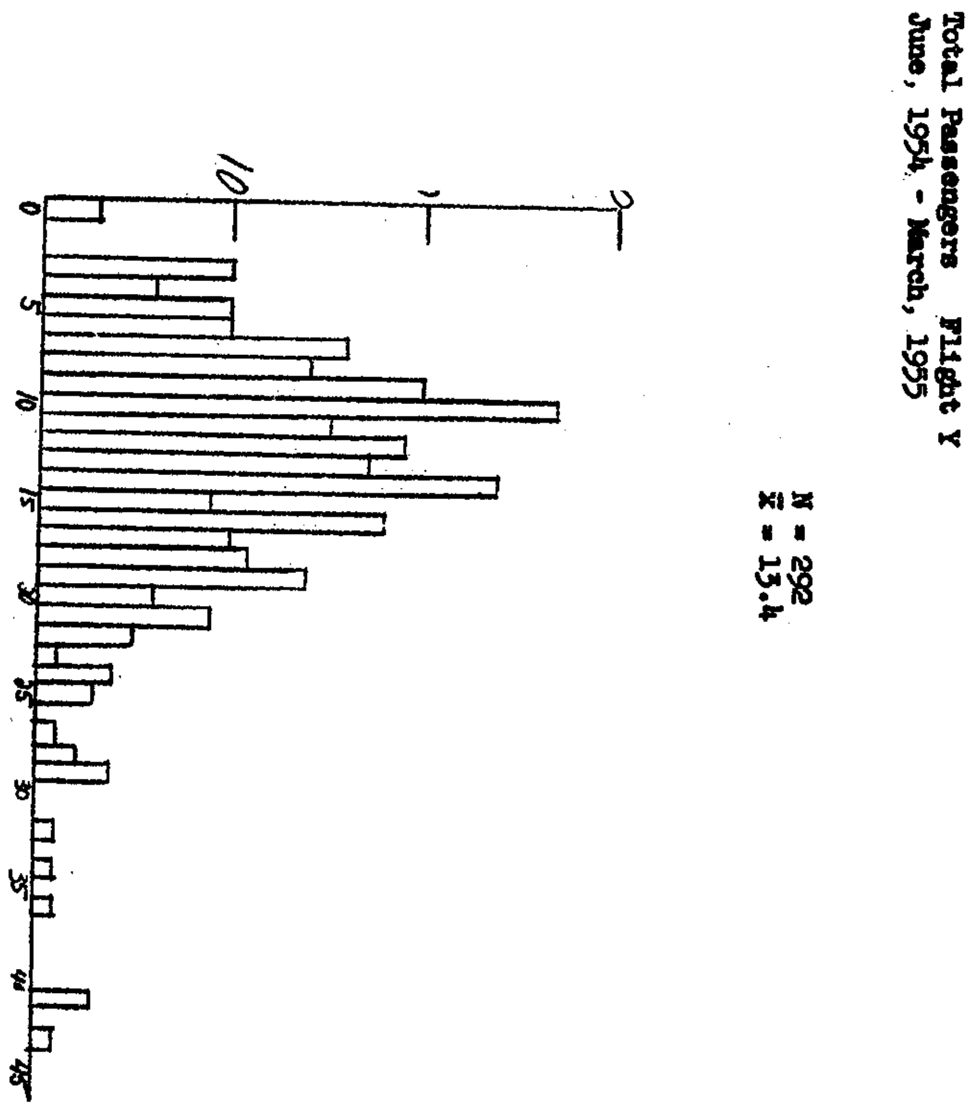
Total Passengers Flight Y
June, 1954 - March, 1955
N = 292
x̄ = 13.4
0
5
10
15
20
25
30
35
40
45
10

Fig. 11.

References

[1] W. Feller: An Introduction to Probability Theory and Its Application, Wiley, New York, 1950.

[2] W. L. Grossman: Air Passenger Traffic, Rheuse Press Division. Chemical Publ. Co., Brooklyn, N.Y., 1947, pp. 125-171.

[3] R. G. Kendall: The Advanced Theory of Statistics. Giffin and Co., London, 1943, Vol. I [pp. 124-125].

[4] W. Vickrey: Some Implications of Marginal Cost Pricing for Public Utilities. Paper presented at the 67th annual meeting of the American Economic Association, Detroit, Dec. 28, 1954.

[5] T. Wolfe: Air Transportation Traffic and Management. New York, McGraw Hill, 1950. pp. 445-460.