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An Analysis of the Demand for Some Machine Repair Parts*

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1. Introduction:

In the mathematical theory of optimal inventory policy under conditions of uncertainty three assumptions lead to crucial simplifications. These are stationarity, independence, and known distribution of demand. By stationarity is meant that the demand distribution and all cost parameters are unchanging over time. Independence means that the demand distribution in any time period is statistically independent of the distributions for all other time periods. If in addition we know the form of the known distribution of demand, in particular that it be Poisson or negative exponential, then, granted certain other restrictions, a complete solution can be given ([1], [2]).

The purpose of this paper is to examine some data on the distribution of demand for machinery repair parts to see if the conditions of stationarity

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Corrections to Cowles Commission Discussion Paper: Economics 2121

p. 4, line 1: Change "open-parameter" to "non-parametric"

p. 5, line 7: Change "non-parameter" to "non-parametric"

p. 7, Table III: Part C should read:

	Expected
Over 249	5.81

and independence are satisfied and if the distribution can be described by one of these two simple distribution functions. The data arise in the following way: a manufacturing company uses machine repair parts in its plants producing its final output. These parts are machined in plants of its own and stocked both in stockrooms at the part-using plants and in a central warehouse at the part producing plants. The part using plants follow an (s,S) two-bin type policy. That is, an order of a given size is placed whenever stocks at these plants fall below a given level. The total of orders placed by these plants per unit of time is the demand of the central warehouse.

Demands at the part using level are generated by failure of parts installed in operating machinery. As is well known, if the conditional probability of failure of a part is independent of its age, the distribution of time to failure is negative exponential and that of failures per unit of time, assuming replacements are made instantly, is Poisson (See for example, Davis [3, pp. 117-118]). Davis examines a large number of sets of time-to-failure data and suggests that "the exponential theory of failure may be regarded as a useful approximation of certain classes of failure distributions" [3, p. 123]. Thus one might expect, as a first approximation, that part usage is distributed according to the Poisson law. But in this case there are many plants, each with many machines, which use these parts. If the Poisson parameter for each is itself a random variable distributed negative-exponentially, then the resulting distribution of part usage is geometric -- the discrete case of the negative exponential distribution (cf. [5, pp. 124-125), which is a convenient approximation to it for some purposes.[1] These considerations, of course, relate to the usage of repair parts in the part using plants. The fact that they send

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1 I am indebted to Martin Beckmann for this suggestion.

orders to the central warehouse only when their stocks fall to certain levels might cause the distribution of demands on the central warehouse to differ substantially from that of part usage in all plants.

In this paper we shall analyze the demand per three months on the central warehouse for six different parts over a seven year period. Central warehouse demand rather than actual usage was studied since data are available in this form only; likewise, three months was the shortest period for which repair part demand was available. Our conclusions are 1) that the distribution of demand for these six parts may be considered to be stationary and independent in the sense defined above, 2) that the Poisson distribution gives a particularly bad fit to these observed distributions, and 3) that the negative exponential distribution gives a much better fit in all cases, although use of the χ^2 criterion would lead us to reject the hypotheses of a negative exponential distribution in two of the six cases.

2. Tests for Stationarity and Independence

Departures from stationarity might be of at least two different types, trend and seasonal variation. The company's sales have been increasing and with more machines of a particular type being used or existing machines being used more intensively it might be expected that more repair parts would be needed as time progresses. Also there is a tendency for sales to be greater in the last two quarters of the year than in the first two so that one might expect that repair part demand would vary correspondingly. We thus wish to test for the existence of trend and seasonal variation.

One way to do so is to use a two way analysis of variance, classifying our observations by year and quarter. Since we can't assume that the observations have come from a normal population we have used Friedman's Ranked Analysis of Variance [4], a standard open-parameter technique. In Table I, below, the values of his χ_r^2 statistic for each of the six parts for the year and quarter comparisons as well as the approximate significance level are given (in the limit χ_r^2 is distributed as χ^2). In each case the tests were based upon four quarterly observations for each of seven years.

Table I

Rank analysis of Variance

Part	χ_r^2 2/	Approx P
A, year	7.205	0.30
quarter	3.643	0.30
B, year	7.688	0.30
quarter	5.229	0.15
C, year	7.714	0.30
quarter	2.486	0.50 (-)
D, year	7.607	0.30
quarter	0.729	0.85
E, year	10.955	0.10
quarter	1.586	0.70
F, year	5.571	0.50 (-)
quarter	8.657	0.05

In only one of the twelve cases is the test statistic significant at the 5 percent level and, if the tests were independent, the probability of at least one "significant result" if there really is no trend or seasonal variation is about 0.46. None of the tests on each part separately would lead us to reject the null hypothesis of stationarity. However, one notices that ten of the twelve P values are less than 0.50. If the twelve tests were independent and if there were no trend or seasonality, the probability of a greater

2/ For the year comparison, χ_r^2 has six degrees of freedom, for the quarter comparison three.

deviation from expectation is about 0.04. This gives us reason for doubting that for all six parts there is no trend or seasonality. But it does not appear from these data that there is a substantial departure from stationarity.

If the distributions of the observations at different times were really statistically independent we would expect to find that the observed demands are uncorrelated. To test for correlation of the observations we have again used a non-parameter test, Spearman's rank correlation coefficient ρ. On the null hypothesis of independence this is distributed approximately as Student's t (Kendall [5, p. 401]). Values of ρ for current demand and demand lagged once and for current demand and demand lagged twice have been

Table II

- Spearman's ρ -

Part	ρ	t	d of f 3/	Approx P
A, one lag	0.171	0.887	26	0.40
two lags	-0.237	-1.221	25	0.20
B, one lag	-0.0979	-0.502	26	0.60
two lags	0.240	1.234	25	0.20
C, one lag	0.123	0.622	25	0.50(+)
two lags	-0.219	-1.101	24	0.30
D, one lag	0.0673	0.337	25	0.70
two lags	0.164	0.814	24	0.40
E, one lag	0.235	1.207	25	0.20
two lags	0.352	1.844	24	0.07
F, one lag	-0.260	-1.345	25	0.20
two lags	-0.0649	-0.319	24	0.70

computed. The results are shown in Table II. None of the ρ's is significant at the 5 percent level. In addition, five of the twelve are negative and four

3/ One more observation was available for parts A and B than for the others.

of the twelve P values are greater than 0.50. It would appear that the data do not refute the hypothesis of independence.

3. Fit of distribution functions

When we turn to the testing of hypotheses about the form of the underlying distribution of demand, the results are much less favorable. For all six parts it would appear that the Poisson distribution fits very badly. For each of them, there is a much greater variation in demand than one would expect if the underlying distributions were Poisson.4/ Because of this we have made no goodness of fit tests for these distributions.

However, the negative exponential5/ appears to fit rather better and such tests have been made for it. The results of these tests are reproduced in Table 3 below. In two cases, Parts A and F, the fit appears to be very good. For two others, C and E, it appears particularly bad while for the remaining two the test statistic is significant at about the 10 percent level. In all cases the distribution fits well in the upper tail. Deviations from expectation are primarily in the lower tail, where too few, and in the center, where too many, demands are observed.

4/ The normal distribution appears to fit badly for the same reason.

5/ This distribution is expressed by $F(x) = 1 - e^{-bx}$, where $\frac{1}{b}$ is the mean of the distribution.

- Table III -

Chi-square Goodness of Fit Tests

Part A*

Demand	Observed	Expected
Under 25	6	5.63
25-49	4	4.39
50-99	7	5.88
100-199	5	5.38
Over 199	2	2.72

$\chi^2(3) = 0.488$

Approx P = 0.92

Part B*

Demand	Observed	Expected
Under 25	4	6.17
25-49	3	4.68
50-99	11	5.98
100-199	4	5.04
Over 199	2	2.13

$\chi^2(3) = 5.793$

Approx P = 0.12

Part C

Demand	Observed	Expected
Under 50	3	7.50
50-99	4	5.54
100-149	6	4.04
150-249	13	5.10
Over 249	2	5.8

$\chi^2(3) = 18.801$

Approx P = 0.001

Part D

Demand	Observed	Expected
Under 10	6	5.58
10-19	2	4.68
20-39	4	6.63
40-59	12	6.75
Over 59	4	4.35

$\chi^2(3) = 6.720$

Approx P = 0.10

Part E

Demand	Observed	Expected
Under 10	13	7.12
10-19	2	5.55
20-39	2	7.07
40-59	9	5.86
Over 59	2	2.40

$\chi^2(3) = 12.497$

Approx P = 0.01

Demand	Observed	Expected
Under 25	5	5.13
25-49	4	4.27
50-99	7	6.30
100-199	7	6.92
Over 199	5	5.39

$\chi^2(3) = 0.128$

Approx P = 0.99

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* Tests for Parts A and B were based upon demand for 24 quarters only since the reported demands for 5 quarters are suspected of being subject to serious error.

Since we have but 24 to 28 observations upon which to base these tests and since the χ^2 test is not a very powerful one, there is serious doubt that the underlying distribution of demand is negative exponential in four of the six cases. But it would appear that this distribution gives the best simple approximation to the observed distributions.[6]

Finally we note that if our null hypotheses is that the distributions for all six parts are negative exponential, and if we assume the tests are independent, the appropriate test statistic is the sum of the individual χ^2's. It is distributed as χ^2 with 18 degrees of freedom. Our observed $\chi^2(18) = 44.427$, which is significant at the 0.001 level. We would thus reject the hypothesis that all the distributions are negative exponential.

References

[1] Martin Beckmann and Richard F. Muth, "On the Solution to the 'Fundamental Equation' of Inventory Theory", CCDP: Econ. No. 2116 (mimeographed).

[2] ________________, "An Inventory Policy for a Case of Lagged Supply", (unpublished).

[3] D. J. Davis, "An Analysis of Some Failure Data", Journal of the American Statistical Association, 47, (June, 1952), pp. 113-150.

[4] Milton Friedman, "The Use of Ranks to Avoid the Assumption of Normality Implicity in the Analysis of Variance", Journal of the American Statistical Association, 32, (December, 1937), pp. 675-701.

[5] M. G. Kendall, The Advanced Theory of Statistics, Vol. I, London, C. Giffen and Co., Ltd., 1948.

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6 In only one of the cases, Part C, was it possible to obtain a better fit with the more general Gamma distribution, of which the negative exponential is a member.