COWLES COMMISSION DISCUSSION PAPER: ECONOMICS NO. 2120

On the Inventory Problem of Beckmann and Muth**

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March 17, 1955

In Beckmann and Muth's paper "On the Solution to the 'Fundamental Equation' of Inventory Theory"[1], the proof of convexity of the function g is faulty and apparently cannot be repaired. It was the convexity of this function which in turn assured that there was an optimal two-bin policy* for the inventory problem.

However, the present paper gives a proof of optimality of the two-bin policy under the assumption that the frequency function f of stock demanded is decreasing. This is possibly the best answer to the question of optimality of the two-bin policy.[2]

In the course of the proof we observe that, in any case, there is always a set -- we call it the "policy set" -- which describes the optimal policy.

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1 CCDP Econ. No. 2116.

* i.e., a policy of the so-called (s,S)-type.

2 Compare Dvoretsky, Kiefer, and Wolfowitz, "On the Optimal Character of the (s,S) policy in Inventory Theory" Econometrica 21 (1953) pp. 586-596, where the optimality of the two-bin policy is fully treated in the one-period case.

** Research undertaken by the Cowles Commission for Research in Economics under contract Nonr-358(01), NR 047-006 with the Office of Naval Research.

When this set reduces to an interval $]s,S[$ the corresponding policy is the two-bin policy with parameters s and S.

§1 - The inventory problem with which we are concerned can be summarized as follows. Denote by $\ell(x)$ (resp. $L(x)$) the expected discounted loss under optimal policy for each of all future periods, at a time just before (resp. after) an order is made to replenish the inventory, the stock presently on hand being x.

It is immediately seen[3] that the following system of equations holds:

$$(\mathrm{I}) \quad \begin{cases} \ell(x) = \min\left[L(x), \inf\limits_{u>0} (K + du + L(x+u))\right], & (0 \leq x < \infty). \\ L(x) = \lambda(x) + \alpha \int_0^{\infty} \ell(\max(0, x - z))\, f(z)\, dz, & (0 \leq x < \infty). \end{cases}$$

where f is the frequency function of the amount of stock demanded in any period; α the discount factor; $\lambda(x)$ the <u>expected</u> value of the shortage penalty (plus storage costs) incurred during the period; K the fixed and k the per-unit, ordering costs.

f is the frequency function for a probability distribution on the positive half-axis $[0,\infty[$. One assumes that $0 \leq \alpha < 1$, $K \geq 0$, $k \geq 0$, and that λ is continuous.[4]

By inventory theory the system (I) has a unique solution for the functions ℓ and L and they are continuous in their range $[0,\infty[$.[5]

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3 See Beckmann and Muth, <u>loc. cit</u>., for a complete definition of the problem.

4 In the Beckmann-Muth paper $\lambda(x) = cx + \int_x^{\infty} [A + a(z-x)]\, f(z)\, dz$.

5 In fact for the existence and unicity of the solution one need not assume that λ is continuous but only that it is measurable and takes bounded sets into bounded sets. (Dvoretsky-Kiefer-Wolfowitz). The continuity of ℓ L, however, depends upon that of λ.

In what follows λ and f will be assumed continuously differentiable as many times as needed.

§2 - The upper stock limit and the policy set

Let us denote[6] by $g(x)$ (resp. $G(x)$) the function $\not{l}(x) + kx$ (resp. $L(x) + kx$). $G(x) \geq 0$ for $x \in [0,\infty[$. We denote by S any point of this interval where G attains its inf, or put $S = +\infty$ if G does not attain its inf. (In spite of the possible abuse of language, $G(S)$ will be used to denote the inf of G in $[0,\infty[$.)

One has, putting $y = x + u$, $L(x+u) + ku = L(y) + k(y - x) = G(y) - kx$. Since L is continuous one thus has for x in the range $[0,S[$ the relations

$$\inf_{u > 0} L(x + n) + ku = \inf_{u \geq 0} L(x + u) + ku$$

$$= - kx + \inf_{y \geq x} G(y)$$

$$= - kx + G(S)$$

It follows that, (1) <u>if the stock level</u> $x \geq S$ <u>then it is optimal not to order anything;</u> (2) <u>if the stock level</u> $x < S$ <u>an optimal policy will not bring the stock level above</u> S: <u>S is the upper stock limit</u>.

We can thus stick to the interval $[0,S[$, and here the system (I) becomes

$$(\mathrm{I}') \quad \begin{cases} \not{l}(x) = \min\,(L(x),\, m - kx) & (0 \leq x < S) \\[2ex] L(x) = \lambda(x) + \alpha \int_0^x \not{l}(x - z)\, d\,F(z) + \alpha\, \not{l}(0)\,(1 - F(x)) & \\ & (0 \leq x < S) \end{cases}$$

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6 The introduction of the new function G(there noted g) and the set A (see below) was made by Beckmann in the course of his (alas, unsuccessful!) proof. (See Beckmann and Muth, *loc. cit.*)

where $m = K + G(S)$ and $F(x) = \int_0^x f(z)\,dz$ is the cumulative distribution function.

Substituting the new functions g and G, (I') becomes in turn

$$(I'')\quad \begin{cases} g(x) = \min\,(G(x), m) & (0 \leq x < S) \\ G(x) = \mu(x) + \alpha \int_0^x f(x-z)\,g(z)\,dz + \alpha\, g(0)\,(1 - F(x)) & (0 \leq x < S) \end{cases}$$

where $\mu(x) = kx + \lambda(x) - k\alpha \int_0^x (x-z)\,f(z)\,dz$.

One sees from (I") that G admits a continuous derivative (noted G') throughout the interval, and

$$G'(x) = \mu'(x) + \alpha \int_0^x f'(x-z)\,g(z)\,dz + \alpha g(x)\,f(0) - \alpha g(0)\,f(x)$$

This becomes, by integrating by parts

$$G'(x) = \mu'(x) + \alpha \int_0^x f(x-z)\,dg(z),$$

where the latter integral is to be interpreted in the sense of Riemann-Stieltjes. Finally, if we denote by A the subset of $[0,S[$ where $G(x) < m$, the latter equation in turn becomes[7] (using I"),

$$G'(x) = \mu'(x) + \alpha \int_{0 \atop A}^{x} f(x-z)\,d\,G(z) + \alpha \int_{0 \atop \complement A}^{x} f(x-z)\,dm$$

(In these integrals the range of the variable is understood to be restricted not only to the interval between the lower and upper bounds, but also to the set written below the integral sign.) Now the second of these two integrals

7 At least if A is a <u>finite</u> union of intervals. (In general A, being an open set, is a countable union of open intervals.)

vanishes, as m is a constant. We thus arrive at the following linear integral equation:

$$(I''') \qquad G'(x) = \mu'(x) + \alpha \int_{0\atop A}^{x} f(x-z)\, G'(z)\, dz, \qquad (0 \leq x < S)$$

This equation is fundamental for what follows.

If the set A is known, then G' (in the range $[0,S[$) is given by I''', because this equation has a unique solution.[8] As A is the set where $G(x) < m$, the function G and the number m are in turn determined up to a constant (the same for each), and therefore (by I") also the function g up to the same added constant.

It follows that the set A itself determines how to act optimally: it describes the best policy.

We shall see in the next § how, under certain assumptions on λ and f, the set A reduces to a single interval $]s,S[$ and that the resulting policy is "when stock falls below s, order so as to bring it up to S." This is the so-called two-bin policy.[9,10]

§3 - Case of decreasing frequency of demand

In this § we shall assume that the frequency frunction f is decreasing ($x \leq y \Rightarrow f(x) \geq f(y)$) in its range $[0,\infty[$, and that the loss function λ, already assumed continuous, is in addition convex.[11] In order to conduct our

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8 Same reasoning as that upon the fundamental inventory equation.
9 Dvoretsky, Kiefer, and Wolfowitz use the term "(s,S) policy". (loc.cit.)
10 I suspect that, more generally, if A is any finite collection of intervals, there is a corresponding "n-bin" policy whose parameters are the endpoints of these intervals.
11 The function λ is continuous and convex in all of the cases studied earlier -- Dvoretsky-Kiefer-Wolfowitz, Bellman, and Beckmann-Muth. See the latter paper for the relevant citations.

proofs, we need to assume moreover that both these functions are continuously differentiable one or more times, as the case may be (an assumption which is probably not essential).

From the convexity of λ the convexity of μ follows, for

$$\mu'(x) = k + \lambda'(x) + d\,\alpha \int_0^x z\, f(z)\, dz$$

and

$$\mu''(x) = \lambda''(x) + k\,\alpha\, x\, f(x) \geq \lambda''(x) \geq 0.$$

The first derivative μ' is thus an <u>increasing</u> function.

The crucial step in out argument is the following.

<u>Lemma 1</u> - <u>Suppose</u> $[0,a]$ <u>and</u> $[a,b]$ <u>are two consecutive intervals, starting at the origin and contained in the basic range</u> $[0,S[$, <u>and such that</u> G <u>is decreasing in the first and increasing in the second. It follows that</u> $G'(a) \leq G'(b)$; <u>moreover</u>, $G'(a) < G'(b)$ <u>unless</u> G <u>is constant in this interval</u>.

From the equation (I''') of §2, one deduces that for each $a \leq x \leq b$,

$$G'(x) - G'(a) = \mu'(x) - \mu'(a) + \alpha \underset{A}{\int_0^x} f(x-z)\, G'(z)dz - \alpha \underset{A}{\int_0^a} f(a-z)\, G'(z)\, dz$$

$$= [\mu'(x) - \mu'(a)] + \alpha \underset{A}{\int_0^a} [f(a-z) - f(x-z)]\, [-G'(z)]\, dz$$

$$+ \alpha \underset{A}{\int_a^x} [f(x-z)]\, [G'(z)]\, dz$$

By hypothesis, each term in brackets in this last expression is ≥ 0, and so $G'(b) - G'(a) \geq 0$. Now suppose $G'(b) = G'(a) = 0$. The two terms

$\mu'(x) - \mu'(a)$ and

$$\alpha \underset{A}{\int_0^a} [f(a - z) - f(x - z)] [-G'(z)] dz$$

are both increasing functions of x in $[a,b]$ and vanish at each end of this interval, hence vanish at all points of the interval, and one has

$$G'(x) = \alpha \underset{A}{\int_a^x} f(x - z) G'(z) dz \qquad (a \leq x \leq b),$$

an integral equation having as solution $G' = 0$, Q.E.D.

<u>Theorem 1</u> - <u>The function</u> G <u>is decreasing throughout the whole range</u> $[0,S[$.

This is seen by considering how the continuously differentiable function $G(x)$ must vary as x runs from 0 to S, in view of lemma 1. In the first case if $G(x)$ starts off at $x = 0$ by <u>increasing</u>, then upon taking $a = 0$ and $b > 0$ but suitably small, we deduce that $G'(x) \geq G'(a) \geq 0$ for $0 \leq x \leq b$ and $G'(b) > G'(a)$: G's growth is accelerating. As G is continuous, repeated applications of this reasoning show, step by step, that G is increasing throughout $[0,S[$, which is absurd, (in view of the definition of S as the minimizer of G) unless $S = 0$, when the theorem is trivially true.

In the second case, G starts falling as we leave the origin. Now if G is everywhere decreasing in $[0,S[$ then the theorem holds; otherwise G decreases for an interval $[0,a[\subset [0,S[$ but then is increasing throughout some suitably small interval $[a,b]$ consecutive to $[0,a]$ (by continuity of G'). Then lemma 1 tells us that $G'(b) > G'(a)$ and we see that G

remains increasing after the point a until we meet S. (Same reasoning as in case one.) But this contradicts the definition of S, Q.E.D.

Corollary - The policy set A is an interval]s,S[and these two numbers are the parameters of a two-bin policy which is optimal.[12]

This conclusion more or less represents the limit of our present state of knowledge about the optimality of the two-bin policy.[13] Taking $\alpha = 0$ yields the earlier result of Dvoretsky-Kiefer-Wolfowitz for one period.

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12 See §2

13 I understand that Bellman's proof attempting to go beyond our present case suffers the same essential defect as Beckmann's.