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Efficient Communication Networks *

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§§1-4, §6, §8 are a re-statement of material already presented in CCDP Econ. No. 2102. The remaining §§ are new, and constitute partial confirmation of the earlier results plus advances into fresh territory.

§1 - The problem

A communication network for a set V of people provides links between certain couples of elements of V. These links are one-way. Information can be sent from a person x to a person y providing there is a path from x to y in the network. The number of links in the path is called the length of the path. A path is called a route when there is no shorter path with the same terminals.

A network is called adequate[1] when each person can reach each other person. One then is concerned with the length of the longest routes; this is the maximum number of steps which a message must made; it is called the

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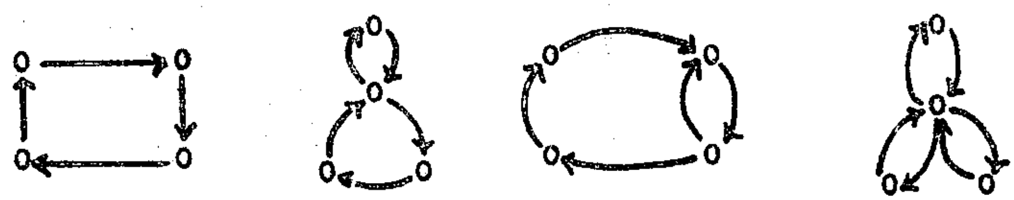
solution time[2] (and denoted T) of the network.

In accordance with ordinary usage, the elements of V are called the <u>vertices</u>.

We are here concerned with the adequate networks which are efficient for the value of T and the number ℓ of links employed. In other words, the problem which we study is, what is the smallest value of T attainable for a given number v of vertices using an adequate network with ℓ links?

To clarify ideas, the following diagrams present four adequate networks where $v = 4$.



$\ell = 4$	$\ell = 5$	$\ell = 5$	$\ell = 6$
$T = 3$	$T = 3$	$T = 3$	$T = 2$

§2 - A characteristic number associated with a network

The first of the above illustrations is simply a circuit; it is the unique adequate network where $\ell = v$. In general $\ell \geq v$. The number $c = \ell - v + 1 \geq 1$ is an important number (called the Betti number) associated with a network: it is the basic number of cycles in the topological space of the network, and is a topological invariant. As the range of ℓ is $v \leq \ell \leq v(v - 1)$, one has $1 \leq c \leq (v - 1)^2$. At the lower extreme $c = 1$ we get the unique cyclic network; at the upper, the unique network (it could be called the "saturated" network) where each pair of persons is linked in both directions. The solution time for $c = 1$ is $T = v - 1$;

for $c = (v - 1)^2$, $T = 1$. For $1 < c < (v - 1)^2$ one has $v - 1 \geq T > 1$, clearly.

Indeed, for $c = v - 1$ there is available the "centralized" network illustrated by the last of the above four illustrations, for which always $T = 2$. It is easy to see that for $v - 1 \leq c < (v - 1)^2$, 2 is the least possible value of T for any network.

Our interest thus lies in the values $1 < c < v$. And here we have necessarily $1 < T < v$ also.

The network R being adequate, each vertex x must have at least one link leading to it ("entrance link") and one link leaving it ("exit link"); x is called a <u>node</u> if there are additional links.

A path each of whose extremities is a node and such that no other nodes occur in it is called a <u>branch</u>. b denoting the number of branches, n the number of nodes, one has $c = 1 - n + b$ (because of the invariance of c).

As any two of the numbers c, ℓ, v determine the third, we will specify them at various times by specifying only two of them.

<u>§3 - Foils - the one-node networks</u>

When $c > 1$, there is at least one surplus link: if we associate to each vertex one of its exit links, there is thus left over $c - 1 \geq 1$ links; choosing one of them, the vertex at its beginning is thus found to have two exits -- it is a node.

A network R with exactly one node has a very simple structure: the node is at the center, from which the other vertices are arranged along branches which begin and end at the node: the graph is a number of "leaves" attached to a central point. The number of these leaves is exactly c. Let us call such a network a "foil".

Thus, in order that an adequate network R have just one node it is necessary that $1 < c < v$. Conversely for each value of c and v such that this inequality holds, there exists a foil which realizes them; it has c leaves.

The solution time of a foil is particularly easy to compute. Define the function σ of a real variable as follows: $\sigma(x + 1) = \sigma(x) + 1$, $\sigma(0) = 0$, $\sigma(x) = \frac{1}{2}$ for $0 < x < 1$. For each integer n, $\sigma(n) = n$; it is a step-function. The least solution time for foils with given ℓ and c is

$$(1) \qquad T_F = 2\,\sigma\left(\frac{\ell - 1}{c}\right) - 1$$

It is obtained by dealing out the vertices around the leaves, and is unique up to a permutation of the vertices. We thus shall refer to it as <u>the foil</u> corresponding to the numbers c and v.

Using the properties of the function σ one obtains rational bounds on its value, and hence of T_F:

$$\frac{2\ell}{c} - 2 \leq T_F \leq \frac{2\ell - 4}{c} \qquad (1 < c < v)$$

These bounds allow a leeway of at most two consecutive values of T. Indeed, (1) shows that T_F is precisely the least integer $\geq \frac{2\ell}{c} - 2$, <u>except</u> when the remainder upon dividing ℓ by c is ≥ 2 and $\leq \frac{1}{2}c$.

§4 - Search for fast networks

It is easy to believe that the foil is the foil is the quickest adequate network. While this is probably true for $v \leq 14$, a counterexample exists for $v = 15$. But the exceptions would at least seem to be scarce.

In any case, the formula $T_v = 2\sigma\left(\frac{\ell-1}{c}\right)-1$ offers an upper bound on the least time for any network. How good is it? Perhaps it is asymptotic (as $p \to \infty$) to the true value? (We shall see more about this later.)

We thus begin a search for quicker networks, which justifies the following classification: let us call a network R <u>fast</u> if its solution time is less than that of all the corresponding foils. (Assuming throughout that $1 < c < v$).

What are the necessary properties of fast networks?

<u>A fast network has three or more nodes.</u>

To see this, let R be a 2-node network, A, B its nodes. There exists a branch going from A to B and vice versa. There may be some leaves attached at A or B. One obtains a new network R' by the following transformation. Any leaves at B are moved to A. While retaining the shortest branch <u>a</u> from A to B and the shortest branch <u>b</u> from B to A, remove all the other branches between the two nodes at the point where they touch B and attach this end to A. It is easy to see that $T' \leq T$, R' has one node, and the same v and ℓ.

<u>If in a network R there is a node with a unique entrance (resp. exit), then there exists a network R' with one less node, same</u> v <u>and</u> ℓ, <u>and such that</u> $T' \leq T$.

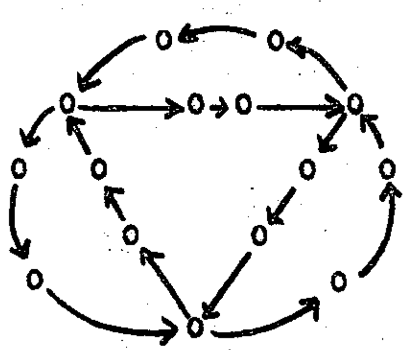
Case of unique entrance: A node B has only one entrance; the unique branch <u>a</u> entering B cannot be a leaf, so leaves from a node $A \neq B$. One obtains R' by leaving intact a route from B to A but moving to A all the tails of the remaining exist branches at B.

We shall show in detail that $T' \leq T$. For this, let r' be a longest route in R', supposed, in order to argue a contradiction, unmatchable by a route in R. r' must then use one of the altered links -- one of the newly created exits at A, -- and hence $A \in r'$. Suppose $B \in r'$: it would have to precede A (for if B follows A, then the unique path a from A to B belongs to r', hence r' visits A twice!). But then as R' has unique path p from B to A, r' = aBpAu, where u is a route in R' from A to some point. But the path r = aBpAu <u>in R</u> is also a route of the same length, contradicting the definition of r'.

On the other hand, suppose $B \notin r'$. One defines a route r in R, r = (part of r' preceding A) AaB (part of r' succeeding A). R is longer than r', which again contradicts the definition of r'. Q.E.D.

Finally, one should note that for a fast network one has $T < \frac{2\ell}{c} - 1$. (For $T \leq T_F - 1 \leq \frac{2\ell - 4}{c} - 1 < \frac{2\ell}{c} - 1$.)

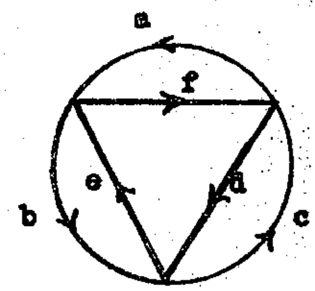
<u>§5 - A fast 3-node network</u>



The network at the left is fast, p = 15, ℓ = 18, c = 4, and the solution time T is 7, one less than T_F. It is interesting to observe that $\frac{2\ell}{c} - 2 = T_F$.

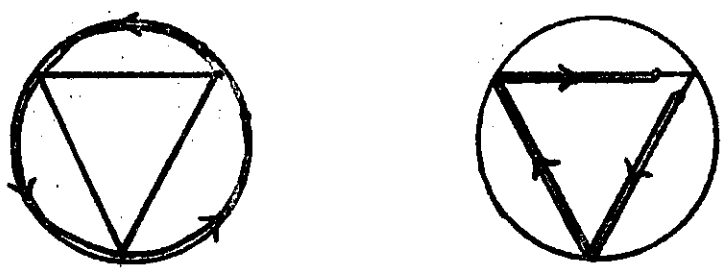
One might hope to find other fast networks by generalizing this network. The first thing to do is to consider any network A with the same oriented graph but with an arbitrary number of people on each of the six branches

A



<u>For the networks A one always has T ≥</u> $\frac{2\ell}{c} - 2$.

The proof of this distinguishes two cases. Case I: at least one of the outside branches, and at least one of the inside branches, is a route between its nodes. Here we can always find two routes r_1, r_2 as indicated in the following figures

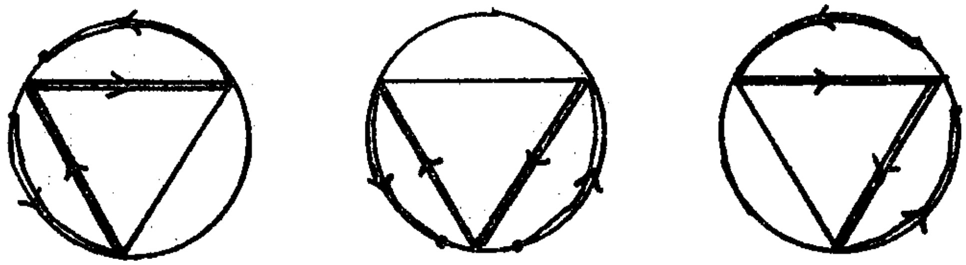


By their construction, $\ell(r_1) + \ell(r_2) = \ell - 4$, hence

$$T \geq \max_i \ell(r_i) \geq \frac{1}{2}[\ell(r_1) + \ell(r_r)] = \frac{1}{2}(\ell-4) = \frac{2\ell}{c} - 2.$$

(We denote by $\ell(r)$ the length of a path r.)

Case II: None of the outside branches is a route between its nodes. Here we can always find three routes r_1, r_2, r_3 as indicated in the following figures:



Therefore $T \geq \max_i \ell(r_i) \geq \frac{1}{3} \Sigma \ell(r_i) = \frac{1}{3}(2\ell - 6) > \frac{\ell}{2} - 2 = \frac{2\ell}{c} - 2.$
Q.E.D.

We pause here to remark,

<u>If in a network</u> R <u>where</u> $1 < c < p$ <u>one has</u> $T \geq \frac{2\ell}{c} - 2$, <u>then</u> $T \geq T_F - 1$.

This results from the "estimate" of the time T_F of the corresponding foil, given in §3.

Continuing our analysis of the networks A, we ask, does T ever equal $T_F - 1$? This could only occur under the conditions of case I of the above proof. Now, by §3, $\ell \equiv 2 \pmod 4$, so that $\frac{\ell}{2} - 2$ is an odd integer, and hence $T = \frac{\ell}{2} - 2$. This equality, in the light of the relations exhibited in Case I, implies that $T = \ell(r_1) = \ell(r_2) = \frac{\ell}{2} - 2$; i.e., $a + b + c = d + e + f = T + 2$.

<u>It follows that each branch of A is a route between its nodes</u>.

To see this, suppose that the branch b of A is not a route. One can then see a route r_1 in A of length $(a - 1) + f + d + (c - 1)$, and a route r_2 of length $(b - 1) + (e - 1)$.

Hence $T \geq \frac{1}{2}[\ell(r_1) + \ell(r_2)] = \frac{1}{2}[a + b + c + d + e + f - 4]$

$= \frac{1}{2}$ [2 (T + 2) - 4] = T. Hence $T = \ell(r_1) = \ell(r_2)$, so b + e = T, and b = d + f, contradicting the hypothesis on b.

All the branches of A are of equal length.

To see this, it suffices to show that $a \leq e$, $b \leq d$, $c \leq f$. By a suitable selection of routes (keeping in mind that each branch is a route), one obtains the inequalities

$$a + f + d - 2 \leq T$$
$$b + e + f - 2 \leq T$$
$$c + d + e - 2 \leq T$$

which imply the desired inequalities, as $T = d + e + f - 2$.

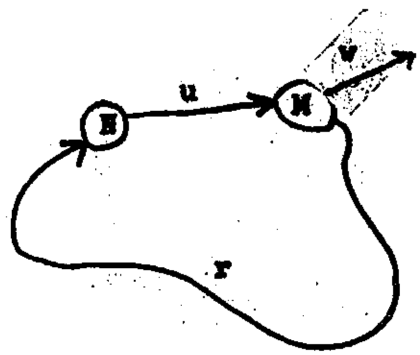
We have demonstrated the following:

In order that a network A with the graph of figure 2 be fast, it is necessary and sufficient that each branch of A have the same length a where a is an odd number ≥ 3. Its solution time T is then equal to $T_F - 1 = \frac{2\ell}{c} - 2$.

Each of the networks A has equal branches. The following theorems show essentially that this property characterizes them.

We call a network node-saturated when there is a branch between each couple of nodes.

Lemma - Let R be an adequate network with $c > 1$ which is not node-saturated. There exists in R a route using four branches.



As R is not node-saturated, there exist nodes M, N with no branch from M to N, Hence there exists a route r from from M to N having ≥ 2 branches.

Suppose that there exist branches u, v such that u enters M, v leaves N, and $u \neq v$. Then urv contains a route using ≥ 4 branches.

Thus we can assume to the contrary that there exists a branch u from N to M with the property that u is the unique exit from N and the unique entrance to M. As M is a node, there exists a branch w leaving M which is not a part of r. The path ruw then contains a route using ≥ 4 branches, Q.E.D.

<u>Theorem 1</u> - <u>The</u> A <u>networks are the only fast networks with equal branches and where each node has at least two entrances</u>.

To prove this, let R be such a network, assumed fast. One then has $b \geq 2n$, i.e. $c \geq n + 1$, i.e., $2c - b \geq 2$. Also $n \geq 3$ and $T \leq T_F - 1 = 2\sigma\left(\frac{\ell-1}{c}\right) - 2$.

R is necessarily node-saturated. For otherwise $4a - 2 \leq T$, therefore $4a \leq 2\sigma\left(\frac{\ell-1}{c}\right)$, i.e. $2a \leq \sigma\left(\frac{\ell-1}{c}\right)$, hence $2a \leq \frac{\ell-1}{c}$ (because a is an integer), i.e. $2\ell c \leq \ell b - b$, i.e. $\ell(2c - b) \leq -b$, i.e. $2c - b \leq -1$, which is absurd.

Thus $c \geq (n-1)^2$ and $T = 3a - 2$, hence $3a \leq 2\sigma\left(\frac{\ell-1}{c}\right)$, and so $3a - 1 < 2\frac{\ell-1}{c}$, i.e. $a(c - 2n + 2) \leq c - 3$. Also, of course, $c < v = 1 - c + a(c + n - 1)$, i.e., $2c \leq a(c + n - 1)$. One thus deduces the following inequalities (where $m = n - 1$):

$$\begin{cases} \frac{2c}{c+m} \leq a \\ a(c - 2m) \leq c - 3 \\ m^2 \leq c \\ 2 \leq m \text{ and } 1 \leq a . \end{cases}$$

Now. this system has as its only integral solutions the following:

(I) $m = 3$, $c = 9$, $a = 2$. $(n = 4, \ell = 24)$

(II) $m = 2$, $c = 5$, $a = 2$. $(n = 3, \ell = 14)$

(III) $m = 2$, $c = 4$, $a \geq 2$. $(n = 3, \ell \geq 12)$

But (I) and (II) violate the stronger condition $3a \leq 2\sigma\left(\frac{\ell-1}{c}\right)$, so that (III) prevails. As R is node-saturated, it is thus one of the A' networks, Q.E.D.

§6 - <u>The conjecture</u> $T \geq \frac{2\ell}{c} - 2$

The study of the class of networks A of the previous § inspires several conjectures. For one thing, the best rational lower bound $\frac{2\ell}{c} - 2$ for the time of the foils is the precise time for the networks A. We found in turn that any network where T is $\geq$ this quantity is at best one unit faster than the foils. Is it generally true that $T \geq \frac{2\ell}{c} - 2$? This would indeed be a very satisfying result. In this § are reported a few skirmishes in the nature of direct attacks, suggestive but not sucessful. In the following §§ a deeper attack is made.

Suppose that a network R can be divided into two subnetworks R_i $(i = 1,2)$ with no links in common and precisely one vertex in common. If $T_i \geq \frac{2\ell_i}{c_i} - 2$ for the subnetworks, it then follows that $T \geq \frac{2\ell}{c} - 2$ for R itself.

One then has $\ell = \ell_1 + \ell_2$, $c = c_1 + c_2$. More generally if one has $\ell - \ell_1 - \ell_2 = c - c_1 - c_2 \geq 0$ then $\frac{\ell}{c} = \frac{\ell_1 + \ell_2 + k}{c_1 + c_2 + k} = \frac{c_1 a_1 + c_2 a_2 + k}{c_1 + c_2 + k}$

where $a_i = \frac{\ell_i}{c_i} \geq 1$, which is a convex combination of a_1, a_2, and 1, and therefore its value is $\leq \max a_i$, i.e. $\frac{\ell}{c} \leq \max \frac{\ell_i}{c_i}$. From the nature of the

R_i one has $T \geq T_i$, hence

$$T \geq \max T_i \geq \max \frac{2\ell_i}{c_i} - 2 \geq \frac{2\ell}{c} - 2.$$

T. Motzkin has shown[3] that this decomposition, performed as far as possible, leads to a unique tree-like decomposition of R into subnetworks R_i no longer decomposable, and that the latter property is equivalent to the condition, for each two vertices $x \neq y$ of R_i there exists two unoriented paths between x and y which have no links or other vertices in common, (a property which is called doubly-connected). (By an unoriented path we mean a path ignoring the orientation of the links.)

The question is thus reduced to the doubly connected networks.

A reduction to the case where there is at most one branch which is not a route is possible.

For if there are two such branches in R, of lengths a, b, upon removing a we get a new network R' where $c' = c - 1$, $T' \leq T$. By induction on c, we can assume that

$$\frac{\ell}{c} > \frac{T+2}{2} \geq \frac{T'+2}{2} \geq \frac{\ell'}{c'} = \frac{\ell - a}{c - 1}$$

Hence $a > \frac{\ell}{c}$, and similarly $b > \frac{\ell}{c}$. But there exists in R a route beginning at the one branch and ending at the other, of length $\geq a + b - 2$ $> 2\frac{\ell}{c} - 2$, a contradiction.[7]

The methods of the proof of theorem 1 yield the following:

Theorem 2 - For a network where the branches are equal and $c \geq n - 1$ one has $T \geq \frac{2\ell}{c} - 2$.

To prove this, suppose first that the network R is <u>not</u> node-saturated. By the lemma of §5, $T \geq 4a - 2$ where a denotes the common branch length. Hence $T - \left(\frac{2\ell}{c} - 2\right) \geq 4a - \frac{2\ell}{c} = \frac{2a}{c}(c - n + 1) \geq 0$.

While if R <u>is</u> node-saturated, then $c \geq (n-1)^2$ and $T = 3a - 2$, so $T - \left(\frac{2\ell}{c} - 2\right) = 3a - \frac{2\ell}{c} = \frac{a}{c}(3c - 2b) =$

$\frac{a}{c}(c - 2n + 2) \geq \frac{a}{c}((n-1)^2 - 2n + 2) = \frac{a}{c}(n-1)(n-3) \geq 0$ for $n \geq 3$, Q.E.D.

Note -- if each node of R has ≥ 2 entrances, then $b \geq 2n$, i.e., $c \geq n + 1$, and so half of the hypothesis of theorem 2 is satisfied.

<u>§7 - The minimal networks of Luce and their applications</u>

A network is called <u>minimal</u> when it is adequate but such that removing any link makes it inadequate (Christie-Luce-Macy[4]).

> Theorems 3-5 below are rehashes of theorems of Christie-Luce-Macy[5]. We draw further consequences from them, stated as theorems 6-8.

Let R be a network and A a subset of the set V of vertices of R. By <u>collapsing A</u> we mean passing to the new network R' obtained by identifying all the vertices of A with a single one of them and removing all the links which have both endpoints in A. R' may not be a network; because it may result that there is more than one link between the same couple of vertices -- this is the only hitch. When R' <u>is</u> a network, A is called <u>collapsible</u>.

<u>Theorem 3</u> - <u>In a minimal network each adequate subnetwork is collapsible; the result of the collapse is again a minimal network.</u>

This is easy to see.

<u>Theorem 4</u> - <u>R being a minimal network, if one sucessively collapses simple circuits at each stage a minimal network is obtained with</u> $\Delta c = -1$, <u>until finally there is obtained the network consisting of a single point</u> $(c = 0)$.

This is immediate from Th. 3. (A circuit is called simple when it does not cross itself; every circuit is a sum of simple circuits.)

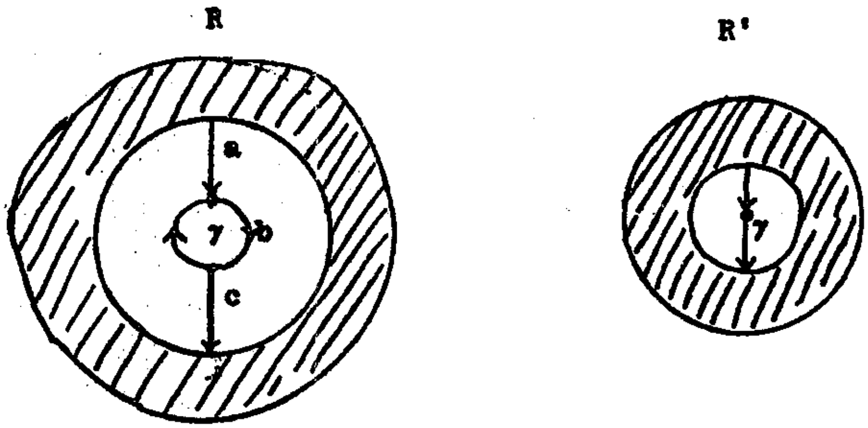
<u>Theorem 5</u> - <u>Each minimal network -- except the network reduced to a single point -- has two or more non-nodes</u>.

One could prove this as follows.[6]

It is trivial for $c = 0$, while for $c > 0$ there exists a circuit γ. By induction on c, the result R' of collapsing γ has two non-nodes.

Now suppose p is a non-node of R', $p \nmid \gamma$ in R'. p is then a non-node in R (else p has 2 exits, say, directly to γ, which violates minimality of R).

Thus the only trouble occurs when γ in R' is itself a non-node of R'. But even here, if γ in R has length ≥ 3, the circuit γ in R has at least one non-node on it, and so it is OK. We are thus left with the case where γ has length 2 in R and is a non-node in R'. But in this case there exists another circuit in R of length ≥ 3 (e.g. any one obtained by traversing links a, b, c and returning, in the diagram below) to which we apply the argument



Theorem 6 - R being an adequate network with one or more nodes, there exists a branch of R which when removed leaves an adequate network.

This because otherwise, if we take the nodes of R as the vertices of a new network R', the branches of R as the links of R', then R' would be a minimal network. But R' necessarily has no non-nodes, contrary to Th. 5, Q.E.D.

Theorem 7 - In each adequate network there exists c independent circuits.

This is clear for $c = 0$ or 1. For $c > 1$, there exists a node, and therefore by Th. 6 a branch u such that the result R' of removing u is adequate. As $c' = c - 1$, by induction on c we conclude that R' has $c - 1$ independent circuits, which are thus independent circuits of R. Now, R' being adequate, there exists in R' a path from the head of u to the tail of u. This path, considered in R, makes up a c^{th} circuit when combined with u, a circuit independent of the preceding as it involves u. Q.E.D.

Th. 6 has also the following corollary, the proof of which is left to the reader.[8]

Theorem 8 - The foils are the only adequate networks where between each couple of persons there is a unique irredundant path.

§8 - The covering conjecture

There is another clue lying in the study of the fast networks of §5 to be used to generate a conjecture. The proofs there proceeded by choosing a set of routes which covered the network. In each case we could have just as well picked 4 routes so that each vertex is used at least twice.

Covering conjecture - R being an adequate network with $c \geq 2$, there exists a set of c routes such that each person in R occurs in at least two of them.

This condition has held up admirably on all the particular networks that the writer is able to sketch on a piece of paper. It is obvious for foils.[9] It is trivial for $c \geq v$ (which is seen by taking a route between consecutive pairs of vertices). It thus naturally limits itself to the range $1 < c < v$. It is appealing in that it is purely combinatorial in nature, making no reference to the lengths of routes, etc. Its value for us lies in this, that

One deduces from the covering conjecture that $T \geq \frac{2v}{c} - 1$.

Otherwise said,[10] $T \geq \frac{2\ell + 2}{c} - 3$. For, n_i denoting the number of vertices on the i^{th} route $r_i (1 \leq i \leq c)$, one has $\sum n_i \geq 2v$. Hence

$$1 + T \geq \max [1 + \ell(r_i)] \geq \frac{1}{c} \sum [1 + \ell(r_i)] = \frac{1}{c} \sum n_i \geq \frac{2v}{c} .$$

The covering conjecture remains unproved. At first sight several avenues of approach are suggested, but one soon discovers roadblocks.[11] In spite of these difficulties, the covering conjecture leads me to the notion of real networks (see below) where its implications are more significant.

In §12 below we shall see how it can be reformulated so that networks, routes, and adequacy all disappear.

§9 - Passage to a continuous problem: Shimbel's real networks.

It is time to consider a more general notion of network, namely that obtained by assigning to each link a positive real number, called its length. An oriented graph where the links have lengths we shall call a real network.[12] The length of a path in such a network is the sum of the lengths of its links. The notion of path is broadened as follows: the endpoints of a path need not be vertices, but can be any point along the length of a link. A route is again a path of minimal length. (There is now an infinity of routes.) The parameter ℓ stands for the sum of the lengths of all the links. (one no longer has the identity $c = 1 - v + \ell$; our three parameters have been given independent lives!) Finally, by the solution time, denoted t, for an adequate network one now means the sup of the lengths of all the routes.

The old networks are subsumed under this broader notion: they are the (real) networks where the links have unit length and there is at most one link from x to y. We will thus refer to them as unitary networks. However, the new notion of solution time is incompatible with the old one -- it is a different notion. As a matter of fact, one has for a unitary network, $T = t - 2$.

If the covering conjecture (§8) is true, then its analogue for real networks in general is also true.

This follows from the fact that in any real network R one can interpolate new vertices on the links in such a way that each route in the new network R' -- which is regarded as a unitary network -- is a route in R. If there exist c routes in R' which twice cover the vertices of R' then by shortening these at each end until they terminate on vertices of R we obtain the desired covering of R.

R' is constructed as follows. Denoting the links of R by $(u_i)_{i\in I}$, and the length of u_i by α_i, as the set I of indices is finite there exists a number $e > 0$ with the following property: for each couple of subsets $J \subset I$, $K \subset I$, such that $\sum_{j\in J} \alpha_j > \sum_{k\in K} \alpha_k$, one has $\sum_{j\in J} \alpha_j - \sum_{k\in K} \alpha_k \geq e$. (Any such e will be called an <u>atomic</u> number for R). Replace R by the "rational" network Q where link u_i has the new length $\alpha_i + \Delta\alpha_i = q_i$, q_i being a rational number approaching α_i within $\frac{e}{3n}$ (where n is the number of elements in I). It follows that if $\sum_{j\in I} \alpha_j > \sum_{k\in K} \alpha_k$ then $\sum_{j\in J} q_j > \sum_{k\in K} q_k$ (because $|\Delta \sum_{j\in J} \alpha_j| \leq \sum_{j\in J} |\Delta\alpha_j| \leq \frac{e}{3}$, etc.)

At this point we have a "rational" network Q where each route of Q is a route of R. By taking a common denominator one finally obtains a unitary network R' with the same routes as Q.

In the above reasoning one also sees that the solution time t of R is within e of the solution time t(Q) of Q, and that in turn $t(Q) = d^{-1}\, t(R) = d^{-1}(T(R) + 2)$, where d is the common denominator used in passing from Q to R'; also that $\ell = \ell(R) = \Sigma\alpha_i$ is within $\frac{e}{3}$ of $\Sigma q_i = d^{-1}\, \ell(R')$. As d can be chosen arbitrarily large and e arbitrarily small, we get as a by-product the following:

<u>Approximation theorem</u> - <u>Let</u> G <u>be an oriented graph such that for each realization of</u> G <u>as a real network one has</u> $t \geq a\ell$; <u>then for each realization of</u> G <u>as a unitary network (by adding dummy vertices) one has</u> $T \geq a\ell - 2$. <u>Conversely, if for each realization of</u> G <u>as a unitary network one has</u> $T \geq a\ell + b$, <u>then for each real realization</u> $t \geq a\ell$.

The covering conjecture implies (§8) that $T \geq \frac{2\ell + 2}{c} - 3$. By the Approximation Theorem one sharpens this to $T \geq \frac{2\ell}{c} - 2$:

<u>The Covering Conjecture implies that</u> $t \geq \frac{2\ell}{c}$ <u>for real networks and</u> $T \geq \frac{2\ell}{c} - 2$ <u>for unitary networks</u>.

Besides enhancing the importance of the covering conjecture, this result also throws some light on the foils, for, considered as real networks, the solution times of the foils achieve the value $\frac{2\ell}{c}$: <u>Subject to the covering conjecture, the foils are the fastest among the real networks</u>.

> In the language of physics, the theory of unitary networks is a discrete case of the more general theory of real networks; in the language of economics, it is the result of introducing indivisibles. In such cases experience teaches us that pathological symptoms are apt to appear. The appearance of the exceptional networks A can probably be regarded in this way, the normal state of affairs being represented by the foils.

<u>§10 - Node-saturated networks</u>

One recalls that the A networks are node-saturated. It can be shown that in a node-saturated network there is a circuit which traverses each branch just once. (In fact, the latter property holds if and only if each node of the network has the same number of entrances as exits.)

In the following let R be a node-saturated network, and assume in addition that each branch of R is a route. Let γ be a circuit in R traversing each branch once. One can pick out a route in any sequence of three consecutive branches of γ; by picking these routes suitably one finds that R can be doubly covered by $d = b - [\frac{b}{3}]$ routes, where b is the number of branches and [x] denotes the integral part of x. R having

n nodes, one has $b = n(n-1)$, $c = (n-1)^2$. It follows that $d \leq c$. <u>Thus the covering conjecture holds for R</u>. One can show, with more careful arithmetic, that for $n \geq 5$ one has $d \leq \frac{4}{5} c$.

The approximation theorem §12 shows that $T \geq \frac{2\ell}{d} - 2$, hence $T \geq \frac{5}{2}\frac{\ell}{c} - 2$ for $n \geq 5$. The condition $c < v$ being equivalent to $\frac{\ell}{c} \geq 2$, one has for $1 < c < v$ the inequality $\frac{5}{2}\frac{\ell}{c} - 2 \geq \frac{2\ell}{c} - 1$, so $T \geq \frac{2\ell}{c} - 1$. It follows (§4) that R is <u>not</u> fast. If $n = 3$, we know that R is not fast unless it is one of the A networks. Finally if $n = 4$, and $1 < c < v$, R is not fast; for if $\frac{1}{4}\ell - 2 \leq T \leq T_F - 1$, then $\frac{1}{4}\ell - 2 \leq T_F - 1$, which implies $\ell \leq 12$, hence (as $c = 9$) $\frac{\ell}{c}$ is not ≥ 2, contrary to hypothesis. We have proved

<u>Theorem 10</u> - <u>The networks</u> A <u>are the only fast networks where each branch is a route and such that for each couple</u> (x,y) <u>of nodes, there is precisely one branch from</u> x <u>to</u> y.

§11 - <u>Loadings</u>

Suppose we have an oriented graph G. Whenever we assign lengths to the links of G we obtain a real network, and this structure in turn distinguishes certain paths of G as <u>routes</u>. Now, although there are an infinite number of real network structures that can be given to G, yet they yield only a finite number of routing patterns. (The number of real network structures is even the potency of the continuum.) It is reasonable to look for a theory of routes which takes this fact into account.

How would people think about routes in an actual communication network? Suppose for example, that one person sends out a message addressed to another. The people at the nodes will have to decide which exit link to use to transmit the message, and they can make this decision on the basis of the address of the message; the non-nodal people have no decisions to make. Thus, the routing pattern of the network could be defined by labelling the exit links at each node with the names of the addressees.

Actually, these labels could just as well be simply associated with the branches leaving the node in question. To each branch u of R we thus associate a set $S(u) \subset V$, the set $S(u)$ being defined as follows: $x \in S(u)$ if and only if there is a route beginning with u and ending at x. $S(u)$ is thus the set of names which will occur as addresses on the messages going over u, assuming that messages are sent along routes. We call $S(u)$ the <u>load</u> on the branch u. We can verify formally that the loads determine the routes; in fact, one has the following relation:

(x) <u>In order that a path</u> p <u>be a route, it is necessary and sufficient that the intersection of all the loads of</u> p <u>be</u> <u>nonvoid</u>.

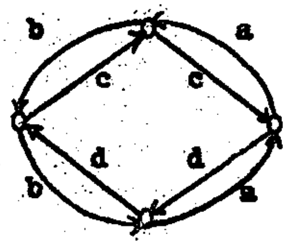
As no circuit is a route, and as R is adequate, one has the following two relations:

(L) <u>The intersection of the loads around a circuit is void</u>.

(A_1) <u>The union of the loads around a node is the whole set</u>.

These two relations are strongly reminiscent of the Kirkhoff laws in electrical circuit theory; in fact the only difference is that there one deals with addition of numbers, while here we have Boolean operations with sets.

The relation (L) can be taken as an axiom. More precisely, suppose that one has an arbitrary set M and a function $u \rightarrow S(u)$ corresponding to each branch of the graph G a set $S(u)$ contained in M. If the axiom (L) is satisfied by the function S, we will call it a <u>loading</u> on the graph G. The relation (x) is then taken as a definition -- it defines the routes corresponding to the loading.



At the left is an example of a loaded graph. The set M consists of the elements a, b, c, d. The routing pattern which results cannot be realized by any real network structure on the same graph.

S being a loading on G, the following notions are of interest: For an arbitrary point A of G one calls the charge (resp. discharge) at A, and notes C(A) (resp. D(A)) the set

$$C(A) = \left[\bigcup_v S(v)\right] \cap \complement \left[\bigcup_u S(u)\right]$$

$$D(A) = \left[\bigcup_u S(u)\right] \cap \complement \left[\bigcup_v S(v)\right]$$

where u runs over the branches entering A and v over the branches leaving A.[13]

One has the following result (depending only on the axiom (L)):

$$\bigcup_A C(A) = \bigcup_A D(A) = \bigcup_u S(u)$$

This equality is proved as follows: Clearly $\bigcup_A D(A) \subset \bigcup_u S(u)$. But suppose, in order to argue toward a contradiction, that there exists an $e \in S(u_o)$ such that e is not in any D(A). One then defines an infinite sequence $u_o, u_1, u_2, \ldots$ of branches, as follows. Having defined

u_k $(k \geq 0)$ such that $e \in S(u_k)$ and $e \notin D(\text{end of } u_k)$, it follows that there leaves from the end of u_k a branch which we denote by u_{k+1} such that $e \in S(u_{k+1})$. This recursive process defines the infinite path $(u_k)_{k \geq 0}$ such that $e \in S(u_k)$ for each k. But as there are only a finite number of branches in G, the sequence must repeat, which violates (L). Analogous argument for C.

An unsolved problem is, to find a convenient set of additional axioms which guarantee adequacy, i.e. the existence of a route between arbitrary points. One such set of axioms, although not perfectly convenient, is (A_1) together with

(A_2) For each node A, $D(A) \neq \phi$

(A_3) For nodes A and B such that $A \neq B$, one has $D(A) \cap D(B) = \phi$.

A simple proof of recursive type shows that with these axioms one has adequacy.

The greatest usefulness of loadings lies in the fact that they transform the notion of route to a local notion, whereas originally it was a global one. The condition of adequacy, originally a global condition, also becomes local, as the axioms $A_1 - A_3$ show.[14]

§12 - The Parcel-handling Puzzle

One stretches the covering conjecture about as far as it can go by conjecturing it for adequately loaded graphs. If we restrict ourselves to those adequately loaded graphs where there is unique route between each couple of vertices, the routing pattern is equally well specified by giving for each couple x,y of vertices the vertex z which comes immediately after x in the unique route from x to y. One also observes that $\sum_x e(x) = \ell$ where x runs over the vertices and $e(x)$ denotes the number of exits at x, and hence $\sum_x [e(x) - 1] = \ell - v = c - 1$. By these steps one arrives at the following conjecture, which has been phrased as a puzzle.

<u>The Parcel Handling Puzzle</u>

There is a group of people who habitually receive parcels. When one of these persons x receives a parcel addressed to y, he relays it to $f(x,y)$. The function f is such that a parcel will eventually come to rest with the person to whom it is addressed, no matter who first receives it.

One sees that a person x for which $f(x,y)$ takes on only a single value as y runs over all the values $\neq x$, is not very important. So let us call the valence of x the number of values assumed by $f(x,y)$ for $y \neq x$, less one. The unimportant people are thus those with zero valence. Denote by V the sum of all the valences in the group.

It is easy to show that, if $V = 0$, a parcel can be addressed and started out in such a way that each person in the group will handle it. On the other hand, can you prove that, if $V \neq 0$, $V + 1$ parcels can be addressed and started out so that each person in the group will handled at least <u>two</u> of these parcels?

One can find other transformations of the covering conjecture.[15]

<u>§13 - The case of two-way links</u>

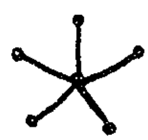
1.- If one considers those networks where the links are <u>two-way</u> it is easy to find all the efficient networks in this class. This change of view corresponds to minimizing the function T over a smaller class of networks, because the networks with two-way links -- we will call them <u>symmetric networks</u> -- can be considered as a subclass of the class of networks studied earlier. (This is done by regarding a symmetric network with ℓ two-way links as a network with 2ℓ one-way links which satisfies the condition "if x is linked to y, then y is linked to x.")

All networks below are assumed symmetric; v denotes the number of vertices, $\ell(R)$ (or simply ℓ) the number of links, k(R) (or simply k) the number of connected components, of R.

One first notes that $v \leq \ell + k$, and that equality holds if and only if the network R is without cycles (i.e., R is a sum of trees, i.e., R is a "forest").

This results from elementary homology theory.

It is clear that R is adequate if and only if it is connected (i.e., k = 1). One then has $\ell \geq v - 1$. Now, being given v vertices, one can link them together as a centralized network:



for which T = 2 and (as there are no cycles) ℓ attains its lower bound $\ell = v - 1$. It follows that $\ell = v - 1$, T = 2, is an efficient point and that the centralized network is efficient. There is thus only one other efficient point -- where T = 1, and here necessarily R is the "saturated" network, where $\ell = \binom{v}{2}$. Indeed, it results that

There are (up to isomorphism) just two efficient symmetric networks for each value of v; they are the centralized and the saturated networks.

To verify this result, it is required to show that a network R with T = 2 and $\ell = v - 1$ is necessarily centralized. Well, R is a tree where any two vertices are at most two steps apart. There exists a vertex r of R with only one link at it. The relation "the path from x to r passes through y" is a partial ordering $y \leq x$ on R for which r is the bottom element and all the elements are arranged in horizontal layers. Clearly r is the only element in its layer, and there are (as $T = 2$) at most two layers above r. If this top layer is void, then R is centralized with r the center; otherwise the top layer is non-void. In the latter case one easily sees that there is just one vertex c in the middle layer and that R is centralized with c as center, Q.E.D.

2. - One can formulate and settle with the same sort of reasoning more complex problems.[16] V being a finite set (the set of vertices for the networks considered), suppose given a set $S \subset V \times V$, and denote, for each network R on V,

$$(1) \qquad T_S(R) = \max_{(x,y) \in S} d_R(x,y).$$

d_R denotes, of course, the distance between x and y for the network R. $T_S(R)$ is the solution time of R when only certain origins and destinations are relevant.

R is said to achieve S when for each $(x,y) \in S$, there is a path in R from x to y. In order for R to achieve S it is necessary and sufficient that $T_S(R) < \infty$.

R is said to be <u>S-efficient</u> when R is efficient for the parameters ℓ and T_S.

One sees that in all these questions S appears only via the function T_S. Now, $d_R(x,x) = 0$ and $d_R(x,y) = d_R(y,x)$. Thus, in view of the definition of T_S, one could assume that $(x,x) \in S$ for each $x \in V$, and also that $(y,x) \in S$ whenever $(x,y) \in S$. Then S is itself simply a symmetric network on V! In what follows S is assumed to satisfy these two conditions.

It follows that, in order for R to achieve S, it is necessary and sufficient that connected components of S form a subpartition of those of R.

As a corollary, one deduces that if R chieves S, then $k(R) \leq k(S)$.

Suppose $S \neq \Delta$ (where Δ denotes the network on V where there are zero links). For a network R, in order that $T_S(R) = 1$ it is clearly necessary and sufficient that $R \supset S$ (i.e., R is obtained from S by suitably adding links to S). In particular as $T_S(S) = 1$, S is always S-efficient. Now suppose R is an S-efficient network $\neq S$. As R in particular achieves S, $k(S) \geq k(R) \geq v - \ell(R)$, it follows that $k(S) \geq v - \ell(R)$, i.e., $\ell(R) \geq k(S)$.

But for the network R_o obtained by re-forming each connected component of S into a centralized network one has $k(R_o) = k(S)$, $\ell(R_o) = v - k(S)$, and $T_S(R_o) \leq 2$.

It follows that, beside the efficient point $\ell = \ell(S)$, $T_S = 1$, there exists at most one other, at $\ell = v - k(S)$, $T_S = 2$. When does the latter point arise? It is clearly necessary for this that $\ell(S) > v - k(S)$, -- i.e., that S is not a forest. This necessary condition is also sufficient. For, S being not a forest, the network $R_o \not\supset S$ (because S possesses a cycle while R_o does not), and hence $T_S(R_o) \neq 1$, i.e., $T_S(R_o) = 2$. We have shown that,

S being an arbitrary symmetric network on V, in order that there exist S-efficient networks R = S on V, it is necessary and sufficient that S possesses a cycle (i.e., that S is not a forest). Then such networks R are necessarily forests with the same connected components as S, and $T_S^{(R)} = 2$. (It follows that R has v - k(S) links). And conversely, this efficient point is realized by each network R obtained by arbitrarily reforming the components of S into centralized networks.

3. - The third problem, more general than each of the preceding, is also settled by the same methods. Let M be a set of messages. By a state of information one means a set $S \subset M \times V$; the relation "$(m,v) \in S$" interpreted "the message m is known to v". By a network one of course understands a set $R \subset V \times V$ such that $(x,x) \in R$ for each $x \in V$. One calls the compose of S and R, and denotes SR, the set of the $(m,v) \in M \times V$ such that there exists $w \in M$ such that $(m,w) \in S$ and $(w,v) \in R$. Similarly for $R_1 R_2$ where R_i are $\subset V \times V$; the compose of two (symmetric) networks is a (symmetric) network; R^n denotes the compose of n R's together.

One supposes that after an initial state S_o of information, each person (vertex) communicates all the messages he knows to each person linked to him. The resutting state of information is $S_o R$. If in each unit time interval this communication process is repeated, the state of information when T = n is $S_o R^n$. (This is a re-phrasing of Shimbel's "fundamental theorem"[17]). Thus, if a desired state of information S_1 is prescribed in order that S_1 be achieved at T = n it is necessary and sufficient that $S_o R^n \supset S_1$.

One denotes by $T_{S_o S_1}(R)$ the least number n such that $S_o R^n \supset S_1$. R is called (S_o,S_1)-efficient when R is efficient for f and T_{S_o,S_1}.

S_o and S_1 being supposed given, we seek the (S_o,S_1)-efficient networks.[18]

The more complex function T_{S_o,S_1} can be reverted to the earlier one as follows. Let (R_i) denote the solutions (or equally well, the minimal solutions) of the relation $S_oR \supset S_1$. It results that for each network R,

$$(1) \qquad T_{S_oS_1}(R) = \min_i T_{R_i}(R).$$

It follows from equation (1) that if R is (S_o,S_1)-efficient, then there exists an i such that R is R_i-efficient and $T_{S_oS_1}(R) = T_{R_i}(R)$. It follows in particular, from §2, that the only possible efficient points are at $T = 1$, $T = 2$. And indeed, we see by §2 that if $T_{S_oS_1}(R) = 1$ then R is one of the R_i and thus is a solution of $S_oR \supset S_1$ with the smallest number of links. On the other hand if $T_{S_oS_1}(R) = 2$ then R is a forest with the same connected components as an R_i; as then $\ell(R) = v - k(R) = v - k(R_i)$, it follows that R_i is a solution of $S_oR_i \supset S_1$ with the largest number of connected components. One is thus let to the following result.

Given initial and terminal states of information S_o, S_1, *on the set* V, *where* $S_o \not\supset S_1$, *and given* ℓ *two-way links with which to connect the elements of* V *into some communication network* R, *the least time* T *of communication achievable is given as follows:*

$$T = \infty \quad \text{for} \quad 0 \le \ell < a$$

$$T = 2 \quad \text{for} \quad a \le \ell < b$$

$$T = 1 \quad \text{for} \quad b \le \ell < \infty,$$

where a *and* b *are two numbers determined by* S_o *and* S_1 *such that* $a \le b$.

Let the network $R = A$ *be a solution of* $S_oR \supset S_1$ *with the largest* number of connected components, $R = B$ a solution with the smallest number of links, One has $a = v - k(A)$, $b = \ell(B)$.

The efficient point at $T = 1$, $\ell = b$, is realized by the network B.

In order that $a = b$, it is necessary and sufficient that there exists a solution of $S_0 R \supset S_1$ which is a forest.

If $a = b$, the efficient networks at $T = 2$, $l = a$, are the forests R with the same connected components as A and such that $T_A(R) = 2$. Each forest of centralized networks where the components are the same as A's then does the trick.

One should note that when $M = V$ and $S_0 = \Delta$ then $A = B = S_1$, which shows that these results are consistent with the earlier ones.

Footnotes

1. A. Shimbel, "Applications of Matrix Algebra to Communication Nets", Bulletin of Mathematical Biophysics 13 (1951) 165-178.

2. Shimbel, loc. cit.

3. Oral communication.

4. L. S. Christie, R. D. Luce, J. Macy, Jr. Communication and learning in a task-oriented group. M.I.T., Research Laboratory of Electronics, Technical report No. 231 (1952). See p. 238.

5. op. cit., pp. 241-242.

6. The writer has not seen Luce's proof.

7. A similar piece of reasoning nearly reduces the problem to the case of unique routes. Suppose that R is a network where $T < \frac{2\ell}{c} - 2$ and where the routes are not unique. This says that there exists two nodes A, B with two different routes r_1, r_2 from A to B. One obtains a new network R' by"collspsing" the two routes r_1, r_2 onto each other so that their respective vertices and links merge two-by-two. We have lost one cycle, so $c' = c - 1$. Also $T' \leq T$ and $\ell' = \ell' - d\,(A,B)$. By induction on c, we can assume that

$$T' \geq 2\frac{\ell'}{c'} - 2.$$

Hence

$$\frac{\ell}{c} > \frac{T+2}{2} \geq \frac{T'+2}{2} \geq \frac{\ell'}{c'} = \frac{\ell - d(A,B)}{c-2};$$

hence $d(A,B) > \frac{\ell}{c}$ in R.

Is it possible to reduce a contradiction from this? If so, we would reduce the conjecture to the case of unique routes.

8. A path is redundant when it passes over a link twice. Theorem 8 is curious; what can be made of it? Its proof is easy, using induction on c and theorem 6.

9. It is straightforward to prove that it holds for all the 2-nodes networks.

10. We deduce in §9 that it actually implies $T \geq \frac{2\ell}{c} - 2$.

11. c being the basic number of circuits in the network, one might hope to find a route corresponding to each such circuit. How to do this?

A simple scheme for a proof of the conjecture is to prove it separately for c even and c odd, and in either case to do so inductively by building up R by adding two branches at a time. This step increases c by 2 so that we can introduce two new routes. We can choose two routes so that we jointly cover each of the new branches twice. But when we go into this in detail, we get into trouble over the unknown effect on previous routes produced by adding new branches.

One might use induction backwards, proving the conjecture in the limit case where there are a given number k of branches between every couple of nodes, and then work backwards by removing two branches at a time. Similar trouble occurs.

The only step which is perfectly clear is this: if R has two branches u,v which are not routes between their nodes, then a covering of the smaller network R' where these branches are removed becomes a covering of R when supplemented by a route from a point of u to a point of v, and by a second route in the opposite direction. This reduces the problem to the networks where there is at most one "redundant" branch.

As leaves are always "redundant" in the above sense, we need never consider networks with 2 or more leaves.

(Heuristically, this suggests that the fast networks will abandon leaves altogether. Is this not a likely conjecture?)

Maybe we are being too highbrow about all this: there may be a direct method of constructing the desired set of c routes. In this case we would only have to check the validity of the construction rather than prove an existence theorem.

12. 1The writer is obliged to A. Shimbel, who kindly made available his unpublished work on real networks.

13. This terminology was chosen with the following interpretation in mind: G is a transportation system, M a set of parcels sent over G, S(u) is the set of parcels that travel over u. A path ρ is a route when some parcel describes it; C(A) (resp. D(A)) is the set of parcels originating (resp. terminating) at A.

14. If loadings are pursued for their own sake, one is carried completely out of the field of networks. For example, a loading on an oriented graph can be formulated as a square matrix $S = (S_{ij})_{(i,j)\in I\times I}$ where the elements S_{ij} are subsets of a set M and the indices run over a set I, satisfying the axiom $S_{ii} = \phi$. (All the sets are assumed finite.) The graph is recovered by taking I as its set of vertices and agreeing that i is linked to j when $S_{ij} \neq \phi$.

For any word $W = i_1, \ldots i_n$ written out of the alphabet I, let us denote by S_W the set $S_{i_1 i_2} \cap S_{i_2 i_3} \cap \ldots \cap S_{i_{n-1} i_n}$. A single axiom suffices to express the condition of an adequate loading.

(AL) <u>For</u> i <u>and</u> j <u>in</u> I, <u>one has</u> $i \neq j$ <u>if and only if there exists a word</u> W <u>such that</u> $L_{iW_j} \neq \phi$.

(It thus appears that adequacy is a converse to the axiom (L).)

15. The following is an empirically-arrived-at conjecture which implies the covering conjecture. (It has, however, a much slimmer empirical base than the covering conjecture; even if it fails, however, something of the same type might hold up.)

Let (S_{ij}) be a square table of subsets S_{ij} of a set M such that $S_{ii} = \phi$ and $R_i \cap C_j \neq \phi$ for $i \neq j$, where R_i (resp. C_i) is the i^{th} row (resp. column) union.

Then, G being the graph where the vertices are the indices i and where i is linked to j when $S_{ij} \neq \phi$, one has either

(a) the connected components of G are simple circuits, or
(b) there exists a sequence $(x_p)_{1 \leq p \leq c}$ of elements of M where the set of indices is [1,c] such that for each i, there are two or more p such that $x_p \in R_i \cup C_i$, where c is the Betti number of G.

The interest in this conjecture is that it is a loading where the axiom (L) has been neglected, hence is of the utmost imaginable simplicity. It clearly implies the covering conjecture.

16. The proofs will only be sketched or indicated briefly. See CCDP Econ. 2117 for full details.

17. Shimbel, loc. cit.

18. Problem posed by R. Radner and A. Tritter in CCDP Econ. 2098.

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