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Two Notes on the Le Chatelier Principle in Linear Programming

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I

Comparative Statics in Linear Programming and the Giffen Paradox[1]

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As Professor Samuelson has pointed out [1] predictions about the directions in which variables tend to change in response to changes in the data can be made also in the case of Linear Programming and without going into the technicalities of linear programming theory. The purpose of this paper is to show how these predictions can be generalized by use of the Minimax (or Lagrange Multiplier) Theorem of Linear Programming.

The point of such qualitative analysis of comparative studies is of course that no explicit representations of the solutions of a linear programming problem as functions of the parameters of the problem are known.

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[1] This supersedes CCDP 2092.

Every change in the data would therefore call for a new analysis of the problem.[2/]

The present generalization of Samuelson's inequality consists in admitting variations of the technology matrix. (section 1) Nothing corresponding to it would seem possible in the case of general (non-linear) constrained maximization problems). It reintroduces the possibility of a Giffen paradox, of which an example is given (section 2).

1. Consider a linear programming problem

$$\max_x b'x$$

subject to

(1) $$Ax \leqq c$$

(2) $$x \geqq 0$$

where $x = (x_k)$ represents the (unknown) activity levels, $b = (b_k)$ the vector of profitabilities, $c = (c_n)$ the vector of capacities and $A = (a_{nk})$ the technology matrix. Let now some arbitrary changes

$$A \longrightarrow A + \delta A$$

$$b \longrightarrow b + \delta b$$

$$c \longrightarrow c + \delta c$$

be imposed, giving rise to changes

$$x \longrightarrow x + \delta x$$

$$\lambda \longrightarrow \lambda + \delta\lambda$$

in the activity levels and "efficiency prices" (Lagrange Multipliers) of the solution respectively.

2/ Of course the general form of the dependence of the solution vector on the data is known. The solution vector is a step function of the profitabilities (the objective function), a piecewise linear function of the capacity vector (the requirements) and a piecewise continuous broken linear function of the coefficients of the technology matrix (the input-output matrix). All this follows from the fact that the optimizing activity levels are the solution of some subset of equations among the constraints.

The minimax theorem of linear programming states the following. Let

(3) $$\phi\,(u,v) = b'u + v'\,(c - Au)$$

Then if $u = x$ is a solution to the linear programming problem and λ is the associated efficiency price vector

(4) $$\phi\,(x,\lambda) = \underset{u \geqq 0}{\text{Max}}\ \phi\,(u,\lambda) = \underset{v \geqq 0}{\text{Min}}\ \phi\,(x,v).$$

Denote the ϕ function associated with the changed problem by ϕ_δ. (4) implies that

(5) $$\phi\,(x,\lambda) \geqq \phi\,(x + \;\; x,\delta\lambda)$$

(6) $$\phi\,(x,\lambda) \leqq \phi\,(x,\ \lambda + \delta\lambda)$$

(7) $$\phi_\delta\,(x + \delta x,\ \lambda + \delta\lambda) \geqq \phi_\delta\,(x,\ \lambda + \delta\lambda)$$

(8) $$\phi_\delta\,(x + \delta x,\ \lambda + \delta\lambda) \leqq \phi_\delta\,(x + \delta x,\ \lambda)$$

Consider the expression

$$(5) + (7) - (6) - (8) \geqq 0$$

Eliminating and rearranging terms one obtains

(9) $$(\delta b' - \lambda'\delta A)\,\delta x - \delta\lambda'(\delta c - \delta A\bar{x}) \geqq 0$$

If $\delta A = 0$, $\delta c = 0$, this inequality is Samuelson's Le Chatelier principle

(10) $$\delta b'\,\delta x \geqq 0$$

as applied to Linear Programming ([1], p. 42, equation (5)). (The inversion of the $\geqq$ sign results from the fact that there a minimum problem is considered). (10) says that activity levels tend to move in the same direction as the precipitating changes in profitabilities. In particular

(11) $$\delta b_j \, \delta x_j \geqq 0$$

in the case of a single change $\delta b_j \neq 0$.

By means of the duality principle a corresponding inequality follows for the efficiency prices.

(12) $$\delta c' \, \delta\lambda \leqq 0$$

stating that efficiency prices tend to move in the opposite direction to the changes in capacities. Combining changes in profitabilities and capacities, we have

(13) $$\delta b' \, \delta x - \delta c' \, \delta\lambda \geqq 0.$$

This says that at least one of the two sets of variables must behave normal: unless activities respond in the proper way to the profitability changes, prices must adjust in the proper way to changes in availability.

The generalization obtained by (9) then consists in the inclusion of changes in the technological coefficients, δA. The result is as follows. Provided the "gross change" δb in profitability is corrected for a simultaneous change in technology in terms of the old prices λ, and provided the gross change in capacities is adjusted for technological changes in terms of the old activity levels $\bar{x}$, then the same tendencies prevail as in the case (13) of mere changes in profitabilities and capacities.

2. As an example consider the case in which only one technological coefficient (say a_{ij}) is changed and all profitabilities and capacities remain constant. From (9) we obtain $-\lambda_i \, \delta a_{ij} \, \delta x_j + \delta\lambda_i \, \delta a_{ij} \, x_j \geqq 0$ or

$$(14) \qquad \delta a_{ij} \, (x_j \, \delta\lambda_i - \lambda_i \, \delta x_j) \geqq 0$$

If either $\lambda_i = 0$ or $x_j = 0$ this inequality is trivial. Let therefore $\lambda_i > 0$, $x_j > 0$. In the case of technological progress, $\delta a_{ij} < 0$, we now conclude that

$$(15) \qquad \frac{\delta x_j}{x_j} \geqq \frac{\delta \lambda_i}{\lambda_i}$$

Suppose that a technological coefficient associated with a "scarce" input and an "active" activity is decreased and that all other constants remain unchanged. Then the induced relative change in the level of the activity that is directly involved, is not exceeded by the relative change in the efficiency price of the input directly involved. In particular a decrease of the activity level calls for at least as large a decrease (in relative size) of the efficiency price.

That an activity level may actually decline in response to a "favorable" technological change is indicated by the fact that Marshall's paradigms of the Giffen paradox may be put into the form of a linear programming problem[3/].

Marshall's example runs as follows:

3/ Prof. H. S. Houthakker has first drawn my attention to this fact.

"I believe that prople in Holland travel by canal boat instead of railway sometimes on account of its cheapness. Suppose a man was in a hurry to travel 150 kilos. [sic]. He had two florins for it and no more. The fare by boat was one cent a kilo, by third class train two cents. So he decided to go 100 kilos by boat and fifty by train: total cost two florins. On arriving at the boat he found the charge had been raised to 1 $\frac{1}{4}$ cents per kilo. 'So then I will travel 133 $\frac{1}{3}$ kilos (or as near as may be) by boat, I can't afford more than 16 $\frac{2}{3}$ kilos by train'" ([2], p. 441).

Let x_1, x_2 denote the distance travelled by train and boat respectively, and let a_1, a_2 be the fare per kilometer. If c_1 denotes the funds available and c_2 the distance to be covered, and if b_1, b_2 are the speeds of the two means of transportation, the object is to minimize

$$b_1 x_1 + b_2 x_2$$

under the conditions

$$\begin{aligned} a_1 x_1 + a_2 x_2 &\leqq c_1 \\ x_1 + x_2 &\geqq c_2 \\ x_1, x_2 &\geqq 0 \end{aligned}$$

In standard form

$$\text{Max} \quad - b_1 x_1 - b_2 x_2$$

$$\text{subject to} \quad \begin{aligned} a_1 x_1 + a_2 x_2 &\leqq c_1 \\ - x_1 - x_2 &\leqq - c_2 \\ x_1, x_2 &\geqq 0 \end{aligned}$$

If, as in Marshall's example $\delta a_2 > 0$ the inequality (13) is reversed

$$(14) \qquad \frac{\delta x_2}{x_2} \leqq \frac{\delta \lambda}{\lambda}$$

Here λ_1 is the efficiency price (or Lagrange Multiplier) associated with the first inequality. Since an efficiency price represents nothing but the marginal productivity of the factor it is associated with, λ_1 expresses the marginal utility (in terms of travel time) of the money sum c_1. The assertion of (14) is therefore, that the distance travelled by boat can increase as a result of a raise of the boat fare only if the marginal utility of money has been increased even more in proportion. This points directly at the source of the Giffen paradox, the income effect of the price change.

II

A Generalized Comparative Statics in Linear Programming

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1. Consider an activity analysis problem $b'x = \max_x$ subject to $Ax \leq c$, $x \geq 0$, and consider some arbitrary changes in the parameters $A \rightarrow A + \delta A$, $c \rightarrow c + \delta c$, $b' \rightarrow b' + \delta b'$, giving rise to corresponding changes in the solutions for activity levels and efficiency prices $\bar{x} \rightarrow \bar{x} + \delta x$ and $\bar{\lambda}' \rightarrow \bar{\lambda}' + \delta \lambda'$. Now since the efficiency price theorem states that if, for any k, $b_k < \sum_n \bar{\lambda} a_{kn}$ then $\bar{x}_k = 0$ (and if $b_k + \delta b_k < \sum_n \lambda_n a_{kn} + \delta \sum_n \lambda_n a_{kn}$ then $\bar{x}_k + \delta x_k = 0$), it is evident that if $\bar{x}_k \neq 0$ and $\bar{x}_k + \delta x_k \neq 0$ then $\delta\ (b_k - \sum_n \bar{\lambda}_n a_{kn}) = 0$, because then $b_k = \sum_n \bar{\lambda}_n a_{kn}$ and $b_k + \delta b_k = \sum_n \bar{\lambda}_n a_{kn} + \delta \sum_n \lambda_n a_{kn}$. Similarly we know that if $b_k < \sum_n \bar{\lambda}_n a_{kn}$ and $b_k + \delta b_k < \sum_n \bar{\lambda}_n a_{kn} + \delta \sum_n \lambda_n a_{kn}$ then $\delta x_k = 0 = \bar{x}_k$.

Hence if <u>both</u> $\delta x_k \neq 0$ <u>and</u> $\delta (b_k - \sum_n \lambda_n a_{kn}) \neq 0$, it must be true either that $\bar{x}_k = 0$ and $\delta x_k > 0$ or that $\bar{x}_k + \delta \bar{x}_k = 0$ and $\delta x_k < 0$. In the first of these two cases $(1 + \delta)\ (b_k - \sum_n \lambda_n a_{kn}) = 0$ and it follows that $\delta (b_k - \sum_n \lambda_n a_{kn}) > 0$; in the second case $b_k - \sum_n \lambda_n a_{kn} = 0$ and it follows that $\delta\ (b_k - \sum_n \lambda_n a_{kn}) < 0$. Hence we have that, for <u>every</u> k,

$$\delta\ (b_k - \sum_n \lambda_n a_{kn})\ \ \delta x_k \ \geq\ 0. \qquad (1)$$

A symmetrical argument, using the restraint $c \geq A\bar{x}$ [also $c + \delta c \geq (1 + \delta)\ A\bar{x}$] and the relation

$$\text{"if } c_n \begin{Bmatrix} > \\ = \end{Bmatrix} \sum_k a_{kn} \bar{x}_k \quad \text{then} \quad \lambda_n \begin{Bmatrix} = \\ \geq \end{Bmatrix} 0 \text{ ",}$$

can be used to prove that

$$\delta\lambda_n \, \delta \, (c_n - \sum_k a_{kn} \bar{x}_k) \leq 0 \qquad (2)$$

for every n. Obviously it follows from these two results that

$$\delta(b' - \lambda' A) \, \delta x \geq 0 \qquad (1^*)$$

and $$\delta\lambda' \, \delta(c - A\bar{x}) \leq 0. \qquad (2^*)$$

2. As a special case consider the change $\delta a_{j_o m_o} < 0$, where all other parameters remain constant. From (1) we have the following equations:

$$(3) \quad \begin{cases} -\sum_n \delta\lambda_n \, a_{kn} \, \delta x_k \geq 0 & \text{for all } k \neq j_o \\ -\lambda_{m_o} \, \delta a_{j_o m_o} \, \delta x_{j_o} - \sum_n \delta\lambda_n a_{j_o n} \, \delta x_{j_o} \geq 0 \end{cases}$$

and from (2) we have

$$(4) \quad \begin{cases} -\sum_k \delta\lambda_n \, a_{kn} \, \delta x_k \leq 0 & \text{for all } n \neq m_o \\ -\delta\lambda_{m_o} \, \delta a_{j_o m_o} \, \bar{x}_{j_o} - \sum_k \delta\lambda_{m_o} \, a_{k m_o} \, \delta x_k \leq 0 . \end{cases}$$

If we subtract the sum over n of the equations (4) from the sum over k of the equations (3) we obtain

$$\delta a_{j_o m_o} (\delta\lambda_{m_o} \bar{x}_{j_o} - \bar{\lambda}_{m_o} \, \delta x_{j_o}) \geq 0 \qquad (5)$$

If $\bar{x}_{j_o}, \lambda_{m_o} \neq 0$, this gives $\frac{\delta x_{j_o}}{x_{j_o}} \geq \frac{\delta \lambda_{m_o}}{\lambda_{m_o}}$, that is either

$\delta\lambda_{m_o} \leq 0$ or $\delta x_{j_o} \geq 0$ or both. In fact this result may be sharpened; after summing (3) if we add the term $\lambda_{m_o} \; a_{j_o m_o} \; \bar{x}_{j_o}$ to both sides of the resulting inequality and rearrange we obtain

$$\delta\lambda_{m_o} \; \delta a_{j_o m_o} \; \bar{x}_{j_o} - \bar{\lambda}_{m_o} \; \delta a_{j_o m_o} \; \delta x_{j_o} \geq \qquad (6)$$

$$\delta\lambda_{m_o} \; \delta a_{j_o m_o} \; \bar{x}_{j_o} + \sum_k \sum_n \delta\lambda_n \; a_{kn} \; \delta x_k .$$

Further, by summing (4) we learn that the righthand term of this inequality is non-negative. (Of course, we know for any set of changes of technical coefficients, with $\delta c = 0$, from $\delta\lambda' \; \delta(c - A\bar{x}) \leq 0$ that $\delta\lambda' \; \delta(A\bar{x}) \geq 0$, of which the result (6) is a special case.)

3. In economic terms this may be interpreted to mean that (a) when factor availabilities do not change the prices of factors and the utilization of factors tends to move in the same direction; (b) this tendency is bounded above by the direct effect on the factor and activity whose transformation ratio changes, or in other words this tendency provides a non-negative lower bound for the direct effect in question. If there is no change in the set of free goods this result is trivial (as indeed it may seem in any case), since for all free factors $\delta\lambda'_f = \lambda'_f = 0$ and for scarce factors $(\delta \, [A\bar{x}])_s = (\delta c)_s = 0$; hence in this case $\delta\lambda' \; \delta(A\bar{x}) = 0$. However it is not at all trivial that since every term in the sum $\delta\lambda' \; \delta(A\bar{x})$ is positive or zero

any change in the set of free goods has the effect of sharpening (5) as indicated in (6).

4. This result may be generalized in terms of the matrix equations (1*) and (2*)

$$\delta\,(b' - \lambda' A)\,\delta x \geq 0 \geq \delta\lambda' \delta\,(c - A\bar{x}).$$

Distributing the operators and rearranging, we obtain

$$(7) \qquad \delta b'\,\delta x + \delta\lambda'\,\delta A\bar{x} - \lambda'\,\delta A\,\delta x \geq \delta\,\lambda'\,\delta(A\bar{x}) \geq \delta\lambda'\,\delta c. \qquad (8)$$

We may now consider some interesting special cases. Consider, for example, $\delta b' = \delta c = 0$, i.e. where the only changes are in technology, this gives

$$\delta\lambda'\,\delta A\,\bar{x} - \lambda'\,\delta A\,\delta x \geq \delta\,\lambda'\,\delta(A\bar{x}) \geq 0, \qquad (6^*)$$

which is the generalization of (6). It should be noted that the rearrangement of terms that was necessary to obtain (7), (8), and (6*) obliterates the term-by-term sharpness of (1*) and (2*) as shown in (1) and (2). Another interesting case is $\lambda A = \delta c = 0$, i.e. when the only changes are in the activity revenues. Here we obtain

$$\delta\,b'\,\delta x \geq \delta\lambda'\,A\,\delta x \geq 0 \qquad (9)$$

which is a sharpened form of the inequality obtained by Samuelson [1].

5. It is of interest to note when it may be expected that the strict inequalities hold, and when not. We have already seen that if there is no change in the set of free factors we may write $\delta\lambda'\,\delta\,(A\bar{x}) = \delta\lambda'\,\delta c$.

(In fact, we may do so also if any factor which ceases to be or becomes free is only just free before or after the parameter change, resp., so that $c_k = A_k\bar{x}$ even though $\bar{\lambda}_k = 0$.) Similarly it can be shown that if there is no change in the set of activities, or if any activity which ceases or begins goes to or from the borderline case of inactivity, then

$$\delta b' \, \delta x = \delta(\lambda' A) \, \delta x.$$

Hence we conclude that if for any factor c_k which ceases to be or becomes free the relation $c_k = \sum_j a_{kj} x_j$ always holds, and if for any activity x_j which ceases or begins the relation $b_j = \sum_k \lambda_k a_{kj}$ always holds, then the relations (7) and (8) may be written

$$\text{(7*)} \qquad \delta b' \delta x + \delta\lambda' \delta A \, \bar{x} - \lambda' \delta A \, \delta x = \delta\lambda' \, \delta(A\bar{x}) = \delta\lambda' \, \delta c. \qquad \text{(8*)}$$

The special cases (6*) and (9) then become

$$\delta\lambda' \delta A \, \bar{x} - \lambda' \delta A \delta x = \delta\lambda' \, \delta\,(A\bar{x}) = 0 \qquad \text{(6**)}$$

$$\text{and} \quad \delta b' \, \delta x = \delta\lambda' A \, \delta x = 0 \qquad \text{(9*)}$$

Thus we know that if only one technical coefficient, and no other parameter, changes and if the consequent adjustment of the system stays within the above limits, then $\dfrac{\delta x_{j_o}}{\bar{x}_{j_o}} = \dfrac{\delta\lambda_{m_o}}{\bar{\lambda}_{m_o}}$; and we know that if the only change is in one activity revenue, and if the consequent adjustment of the system stays within the above limits, then the activity level of the activity whose revenue changes will not change.

Conversely, we may conclude that the necessary and sufficient condition that inequalities hold in (6) and (7), respectively, is that the adjustments attendant upon the changes in parameters must include at least one change in the set of free factors and at least one change in the set of zero activity levels such that the factor in question is interior to the set of free factors when free and the activity level in question is interior to the set zero activity levels when zero.

[1] Samuelson, Paul A., Comparative Statics in the Logic of Economic Maximizing, Rev. of Ec. Studies, Vol. XIV, 1, (1946-47), pp. 41-43.

[2] Letter from A. Marshall to F. Y. Edgeworth, 22. IV. 09. Pigou, A. C. (ed.) Memorials of Alfred Marshall, London 1925, pp. 439-442.