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On The Communications Problem in Airline Reservations[1/2/]

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1. A common feature in the communication rules of airlines concerning reservations is that reservations may be made by any office of the airline (and sometimes also by certain offices of certain other airlines) without further inquiry, until "stop sales" orders are received from the control center in charge of reservations for that flight. From then on seats may be sold only upon requesting and receiving confirmation from the control center. This rule, which has been arrived at after considerable experimentation, means that in order to save communications cost a certain risk of overselling a flight is incurred -- because of the lag between sales, receipt of sales reports at the control center and the receipt of the stop sales order by the sales offices. (This risk is increased when some sales offices that do not generate many sales for a particular flight,

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are not included among the recipients of stop sales orders). It is the purpose of this paper to analyse the nature of the underlying decision problem in terms of a simple model. While the problem concerns optimal rules for communication in a team, there will be no need for explicit reference to and distinction between the various team members. The significance for the theory of organization of this problem is thus severely limited. On the other hand we shall obtain the solution of this particular problem in terms of a rather simple rule. In this rule observable quantities and valuations enter. The existing data, if made available to us, should permit estimation of the valuations that have influenced decisions in the past. Would it be interesting to study such data on decisions under uncertainty? Questions that one might ask are: how consistently have these (implicit) valuations been? Would the decision makers agree with these values if they were revealed to them?

2. In this model we take into consideration those variables only that leave some scope for action. We disregard last minute eventualities, that is to say, no-shows and misconnections. The losses to the airline then consist of three parts: unused seat capacity, penalty for oversold seats, and communication costs. Let these losses per unit be denoted by g, h, k respectively. The controlled variable with respect to which the expected value of loss is to be minimized is the safety cushion c, or equivalently the critical sales level $C - c$, where C denotes (payload) seating capacity. During the interval between the time when sales reach the critical level $C - c$ and the stop sales order is received by all agents further sales take place to the amount e (= excess). Until departure time let there be q cancellations and a demand x for further seats on the flight. Then the loss from unused capacity amounts to

$$g \cdot (c + q - e - x) \quad \text{if this is non-negative,}$$
$$0 \quad \text{otherwise}$$

The loss from oversold seats equals

$$h \cdot (e - c - q) \quad \text{if this is non-negative, and}$$
$$0 \quad \text{otherwise}$$

(For it is clear, that when oversales before receipt of stop sales orders exceed the safety cushion plus cancellation, no additional sales are made).

Finally the cost of communication equals

$$k\,(x + q).$$

Using the symbol $\varphi(y) = \left\{\begin{matrix} y \\ 0 \end{matrix}\right\}$ if $y \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} 0$ we obtain the following expression for the loss as a function of c

$$\text{(1)} \qquad L(c) = g\,\varphi(c + q - e - x) + h\,\varphi(e - c - q) + k(x + q)$$

Now x and q are themselves dependent on c, while without great loss of realism and for the sake of simplicity e may be regarded as independent of c. We shall therefore introduce new variables

- X gross demand for seats until departure time exclusive of cancellations
- Q total cancellations until departure
- $r(c)$ cancellations made before net sales (= gross demand minus cancellations) reach the level $C - c$.

3. While X and Q are random variables, we shall assume $r(c)$ to be of such small variance that it may be regarded a deterministic function of c. This assumption is tolerable in view of the airline experience, that cancellation

fall within rather narrow limits of a definite relationship to net sales. The following identities hold by definition

$$q = Q - r(c) \qquad x = X - r(c) - (C-c) - e$$
$$= X - r(c) + c - e - C$$

and (1) becomes

$$(2) \qquad L(c) = g\,\varphi(C + Q - X) + h\,\varphi(e - Q + r(c) - c) + k\,(X + Q - 2r(c) + c - e - C).$$

Minimizing the expected loss:

$$(3) \qquad \operatorname*{Min}_c L(c) = g\,E\,\varphi\,(C + Q - X) + k\,E(X + Q - e - C) + \operatorname*{Min}_c \left\{ h\,E\,\varphi(r(c) - c + e - Q) + k\,(c - 2r(c) \right\}.$$

For easy reference the term under the Min operator will be denoted by $\not{l}(c)$. A necessary condition for $L(c)$ to be minimal is therefore that

$$\frac{d\not{l}}{dc} = \frac{d}{dc}\,h\,E\,\varphi(r(c) - c + e - Q) + k(1 - 2r'(c)) = 0.$$

4. Let the joint probability distribution of e and Q have a density $p(e, Q)$. Then the first term of $\not{l}(c)$ may be written

$$h \int_{-\infty}^{\infty} d\,Q \int_{0}^{\infty} du\,.\,u\,p(Q - r(c) + c + u, Q)$$

Assuming that p is well behaved so that differentiation under the integral sign and integration by parts are permissible we obtain for its derivative

$$\frac{d}{dc} = h \cdot (r'9c) - 1) \cdot \int d\,Q \int_0^\infty du \cdot u \cdot \frac{\partial p}{\partial e} (Q + r(c) - c + u, Q)$$

$$= - h \left(1 - r'(c)\right) \int d\,Q \int_0^\infty d\,u\,p\,(Q + r(c) - c + u, Q)$$

$$= - h \left(1 \;\; - r'(c)\right) \cdot pr\,(e - Q > c - r(c)\;)$$

where pr denotes probability.
Therefore the minimum condition becomes

$$(4) \qquad pr\,(e - Q > C - r(c)\;) = \frac{k}{h}\,\frac{1 - 2\,r'(c)}{1 - r'(c)}$$

Reintroducing e and q in the left hand side

$$pr\,(e - c - q > 0) = \frac{k}{H}\left(2 - \frac{1}{1 - r'(c)}\right)$$

But $1 - r'(c) = \frac{d}{d(C-c)}\,(C - c + r(c))$ = the rate of increase of gross sales (at the critical sales level) with the critical sales level. And $pr\,(e - c - q > 0)$ = probability of having oversold the plane at departure time.

A rough impression of the behavior of sales and cancellation suggests that the rate of increase of gross sales with net sales is itself mildly non-decreasing with net sales. Therefore the right hand side of (4) is a non-increasing function of c. The left hand side decreases sharply with c, since the probability of overselling grows rapidly as the safety cushion is reduced in size. Although both sides of (4) are thus non-increasing functions of c (4) will have only one solution c. That this yields a minimum is clear from the character of the problem.

5. We have thus obtained the following simple rule: In order that total loss be minimized, the critical sales level must be chosen such that the probability of overselling the flight equals the ratio of communication cost (per message) and penalty (per passenger oversold) times a factor equal to two minus the reciprocal of the rate at which gross sales (at the critical level) increase with that level.

Since the rate of increase of gross sales with net sales exceeds one, its inverse is less than one. But because communication cost k is always small relative to the penalty h for overselling a seat, the expression on the right hand side of (4) falls within the required bounds for a probability.

It is proposed to investigate empirically the frequencies of overselling planes, and the accumulated cancellations as a function of the accumulated net sales, in order to determine what the implicitly held values of $\frac{k}{h}$ are and how consistently they occur.