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# Non-Negative Lagrangean Saddle-Points Without Assumption of Differentiability

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<u>Non-Negative Lagrangean Saddle-Points Without Assumption of Differentiability</u>*

1.1 In [29] Slater deals with the relationship between the existence of saddle points of Lagrangean expressions and a certain type of maximality.** There are two distinguishing features in Slater's treatment of the problem. First, instead of assuming certain functions to be concave, he endows them with a somewhat weaker property. Second, he makes no differentiability assumptions whatsoever. Since it is the latter feature that is of interest to us at present, we shall assume in what follows for the sake of simplicity that the relevant functions are concave, instead of weakening this assumption as was done by Slater. Hence when we speak of "generalizing" Slater's results, it should be understood that this is done with a slight strengthening of postulates where concavity of functions is assumed.***

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* The present paper may be regarded as section IX of the author's "Programming in General Spaces, CCDP. No. 2109, (referred to subsequently as PGS). The numbers in brackets refer to the PGS bibliography or to the bibliography at the end of the present paper. The notations, terminology and basic concepts are those of PGS unless the contrary is specified.

** Actually, Slater speaks of minimization, but from our viewpoint it is more convenient to deal with maximization since other results are so formulated. Also, we use f where he has g and vice versa.

*** We could easily have retained Slater's milder assumptions, since his proof goes through almost verbatim. (This may be the proper place to acknowledge that many proofs are almost identical with those of Slater's; they are included for the sake of completeness. A part of another proof is closely patterned after [19.1].)

1.2 We shall not deal here with the question of circumstances under which the Lagrangean saddle-point implies maximality, since this has been treated in PGS without the utilization of differentiability. We shall only be interested in the existence of such a saddle-point when maximality is assumed and, furthermore, we shall only consider <u>scalar</u> maximization, since the vectorial case was treated in PGS without the assumption of differentiability.

1.3 Consider now the topological linear spaces $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ where $\mathfrak{Y}$ is assumed to be the real axis ($R_1$), and let $f$ and $g$ be concave* continuous functions with domains in $\mathfrak{X}$ and ranges in $\mathfrak{Y}$ and $\mathfrak{Z}$ respectively; also let $X$ be a convex subset of $\mathfrak{X}$ and $P_z$ a convex cone** in $\mathfrak{Z}$. We write $z' \geqq z''$ to mean $z' - z'' \in P_z$.

We say that $x_o$ <u>maximizes</u> $f(x)$ <u>subject to the constraints</u> $g(x) \geqq 0_z$ and $x \in X$ if and only if

$$x_o \in X,$$

$$g(x_o) \geqq 0_z,$$

and

$$f(x) \leqq f(x_o) \quad \text{if} \quad x \in X,\ g(x) \geqq 0_z.$$

The Lagrangean expression***

$$\phi(x, z^*) \equiv f(x) + z^* [g(x)]$$

is said to have a <u>non-negative saddle-point at</u> $(x_o, z^*_o)$ if and only if****

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* Let $\mathcal{U}$, $\mathcal{V}$ be linear systems. A function $h$ with a convex domain in $\mathcal{U}$ and range in $\mathcal{V}$ is said to be <u>concave</u> if and only if $h[\theta u' + (1-\theta) u''] \geqq \theta h(u') + (1 - \theta) h(u'')$ for all $u', u''$ in the domain and all $0 < \theta < 1$. The inequalities are defined in terms of a convex cone, like those in terms of $P_z$ below.

** I.e., if $z \in P_z$ and $\lambda \geq 0$, then $\lambda z \in P_z$; if $z', z'' \in P_z$, then $z' + z'' \in P_z$. $P_z$ if a generalization of the non-negative orthant.

*** $z^*$ denotes a continuous linear ('linear' means additive and homogeneous) functional on $\mathfrak{Z}$.

**** $z^* \geqq 0^*_z$ (or $z^* \geqq 0$) means $z^*(z) \geqq 0$ for all $z \geqq 0_z$.

$$x_o \in X,$$

$$z_o^* \geqq 0,$$

and

$$\phi(x, z_o^*) \leqq \phi(x_o, z_o^*) \leqq \phi(x_o, z^*)$$

for all $x \in X$ and all $z^* \geqq 0$. If $P_z$ has interior points, we write

$$z > 0$$

to mean that $z$ is an <u>interior point</u> of $P_z$.

Slater's relevant result may now be stated as follows. Let $\mathfrak{X}$ and $\mathfrak{Z}$ be finite-dimensional Euclidean spaces, $X$ and $P_z$ their respective non-negative orthants, and assume that, $f$ and $g$ being concave* and continuous, for some $x_* \in X$, $g(x_*) > 0$.** Then if $x_o$ maximizes $f(x)$ subject to $g(x) \geqq 0$ and $x \in X$, the Lagrangean expression has a non-negative saddle point at $(x_o, z_o^*)$ where $x_o$ is the maximizing value (and $z_o^* \geqq 0$). the requirement that $g(x_*) > 0$ for some $x_* \in X$ cannot be dispensed with, as shown by Slater's example with both $\mathfrak{X}$ and $\mathfrak{Z}$ one-dimensional, $f(x) = x - 1$, $g(x) = -(x-1)^2$. This example may be slightly modified to show that $g(x_*) > 0$ cannot be replaced by $0 \neq g(x_*) \geqq 0$. We only need to consider the case where $\mathfrak{X}$ is one-dimensional, $\mathfrak{Z}$ two-dimensional, $f(x) = x - 1$, and

$$g(x) = \begin{pmatrix} -(x-1)^2 \\ -x+2 \end{pmatrix}.$$

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* See above concerning the slightly milder postulate used by Slater. It is known that it is not sufficient to require that $f$ and $g$ be quasi-concave.

** In PGS, we call $g$ <u>Slater-regular</u> if such an $x_*$ exists.

1.4 The present generalization leaves unchanged the properties of $f$ and $g$. I.e., they are assumed continuous and concave, and $g$ is assumed Slater-regular (i.e., $g(x_*) > 0$ for some $x_* \in X$). However, $X$ is any convex set, $P_z$ any convex cone and less is required with regard to $\mathfrak{X}$ and $\mathfrak{Z}$. $\mathfrak{Z}$ is assumed to be linear normed*, while $\mathfrak{X}$ is a topological linear space satisfying the requirement that in its relative topology any line** in $\mathfrak{X}$ be homeomorphic to the Euclidean line under the "natural" isomorphisms.*** The assumption of normability of $\mathfrak{Z}$ is, of course, rather restrictive, but it appears that a relaxation of this assumption, if at all permissible, would call for the generalization of existing results on regular convexity.

2. We shall find it convenient to start by giving a statement of the major theorem to be proved.

<u>Theorem</u>

Let $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ be spaces with the following properties:

(A) $\mathfrak{X}$ is a topological linear space such that in its relative topology any line in $\mathfrak{X}$ is homeomorphic to the Euclidean line under the "natural" isomorphism.

(B) $\mathfrak{Y}$ is the set of reals (i.e., $R_1$).

(C) $\mathfrak{Z}$ is a linear normed space.

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* $\mathfrak{Z}$ was assumed Banach (i.e., linear normed <u>complete</u>) in an earlier version of this paper.

** I.e., a set of points of the form $x_1 + \alpha x_2$ with $x_1, x_2$ fixed elements of $\mathfrak{X}$ and $\alpha$ ranging over all finite reals.

*** That a Hausdorff linear space has this property follows from Tychonoff's theorem (p. 769, [31]) since a linear subspace of a Hausdorff linear space is itself Hausdorff linear in its relative topology (cf. Hyers, [17.1]). Cf. also Stone, [29.1], p. 3-17. Indeed, it is sufficient that $\mathfrak{X}$ be a topological affine space (cf. Fréchet, [9.1], p. 203, and Klee, [18], p. 446). In an earlier version of this paper $\mathfrak{X}$ was assumed locally convex Hausdorff linear.

(D) $P_z$ is a convex cone in $\mathfrak{Z}$. $P_z$ has interior points. (We write $z' \geqq z''$ to mean $z' - z'' \in P_z$; $z' > z''$ to mean[*] $z' - z'' \in \text{Int}\ P_z$.)

(E) $X$ is a (fixed) convex subset of $\mathfrak{X}$.[**]

(F) $f$ is a concave continuous function on $X$ to $\mathfrak{Y}$; $g$ is a concave continuous function on $X$ to $\mathfrak{Z}$.

(G) There exists a point $x_*$ such that

(1') $x_* \in X$

(1") $g(x_*) > 0$.

Let $x_0$ maximize $f$ subject to $x \in X$, $g(x) \geqq 0$. [I.e.,

(2') $x_0 \in X$

(2") $g(x_0) \geqq 0$,

(3) if $x \in X$, $g(x) \geqq 0$, then $f(x) \leqq f(x_0)$.]

Then there exists a functional $z_0^*$ such that

(4) $z_0^* \in \mathfrak{Z}^*$ ($z_0^*$ is linear and continuous[***]),

(5) $z_0^* \geqq 0$ ($z_0^*$ is "non-negative"[****]),

and, writing, for any $x \in \mathfrak{X}$, $z^* \in \mathfrak{Z}^*$,

(6) $\phi(x, z^*) = f(x) + z^*[g(x)]$,

the following (non-negative saddle-point) inequalities hold:

(7) $\phi(x, z_0^*) \leqq (x_0, z_0^*) \leqq \phi(x_0, z^*)$

for all $x \in X$, and all $z^* \geqq 0$.

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* Int A denotes the interior of A.

** Usually X is chosen to be a convex cone in $\mathfrak{X}$, viz. the "non-negative orthant".

*** I.e., it is additive, homogeneous, and bounded.

**** I.e., $z_0^*(z) \geqq 0$ if $z \geqq 0$.

3. Proof of the theorem.

(Note: In what follows it will be assumed throughout that all $x$'s mentioned are elements of $X$. In particular, the phrase "for all $x$" will mean "for all $x \in X$." $z^*$ is always an element of $\mathfrak{Z}^*$.)

3.1 In view of (6), the theorem will have been proved if we find a $z_0^* \geqq 0$ such that

(8') $$f(x_0) + z_0^* [g(x_0)] \geqq f(x) + z_0^* [g(x)] \quad \text{for all } x,$$

and

(8") $$z_0^* [g(x_0)] \leqq z^* [g(x)] \quad \text{for all } z^* \geqq 0.$$

(8") can be rewritten as

($8_1''$) $$(z^* - z_0^*) [g(x_0)] \geqq 0 \quad \text{for all } z^* \geqq 0.$$

Note that, because of (2") and the non-negativeness of $z_0^*$,

(9) $$z_0^* [g(x_0)] \geqq 0.$$

Letting $z^* = 0_z^*$ in ($8_1''$), we obtain

(10) $$- z^* [ g(x_0)] \geqq 0.$$

From (9) and (10) we obtain

(11) $$z_0^* [g(x_0)] = 0.$$

On the other hand, suppose $z_0^* \geqq 0$ and (11) holds. Then (8") clearly follows because of (2"). Hence, it will be sufficient (as well as necessary) to establish the existence of a $z_0^* \geqq 0$ such that (8') and (11) are satisfied.

3.2 It is clear that we may, without loss of generality, assume

(12) $$f(x_0) = 0.$$

(This can be seen if one considers the problem of maximizing $f_1(x) = f(x) - f(x_0)$ which is clearly equivalent to that of maximizing $f(x)$. If one does not wish to postulate (12), one may interpret all the subsequent expressions $f(x)$ to stand for $f_1(x)$.)

We may now restate the desired result as follows:

(T') Let

(13.1) $$g(x_o) \geqq 0,$$

(13.2) $$f(x_o) = 0,$$

and

(14) $$g(x) \geqq 0 \quad \text{implies} \quad f(x) \leqq 0;$$

then there exists a $z_o^*$ such that

(15) $$z_o^* \geqq 0,$$

(16) $$z_o^* \, [g(x_o)] = 0$$

and (using (16) and (13.2) in (8'))

(17) $$f(x) + z_o^* \, [g(x)] \leqq 0 \quad \text{for all } x.$$

Furthermore, in the preceding statement (T') one may delete (16), since it follows from the remainder of (T'). In fact, putting $x = x_o$ in (17), and using (13.2), we get

(18) $$z_o^* \, [g(x_o)] \leqq 0.$$

On the other hand, (9) holds because $z_o^* \geqq 0$ and (2") holds. Hence (16) follows.

Hence we must only prove the proposition (T") obtained from (T') by deleting (16). Now, since (13) follows from our hypotheses, it will be sufficient to prove the following proposition:

(T"'): If

(14) $$g(x) \geqq 0 \quad \text{implies} \quad f(x) \leqq 0,$$

then there exists a $z_o^*$ such that

(15) $$z_o^* \geqq 0$$

(22) $$z^*_{x_1,1} \geq 0,$$

and

(23) $$z^*_{x_1,1}\,[g(x_*)]\; f(x_1) - z^*_{x_1,1}\,[g(x_1)]\; f(x_*) \leq 0.$$

3.3.1 <u>Proof of Lemma 1'</u>.

a. By eq. (14) and Assumption G, we have

(24) $$f(x_*) \leq 0.$$

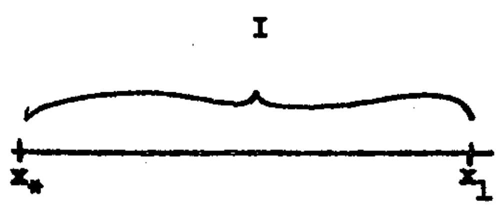
b. Consider now the segment $[x_*, x_1]$, i.e., the set

(25) $$I = \left\{x\colon x = \theta_x\, x_* + (1 - \theta_x)\, x_1,\ 0 \leq \theta_x \leq 1\right\}.$$

We shall write

(26) $x' \leq x''$ (or $x'' \geq x'$) if and only if $x' \in I$, $x'' \in I$ and $\theta_{x'} \geq \theta_{x''}$.

Graphically, we think of $I$ as follows




Thus $x'' \geq x'$ means that both $x'$ and $x''$ are in $I$ and $x''$ is to the <u>right</u> of $x'$ (closer to $x_1$). $x'' > x'$ means $x'' \geq x'$ and $x'' \neq x'$.

c. We now show that there exists an element $\bar{x} \in I$ such that

(27.1) $$f(\bar{x}) = 0$$

(28.2) $$f(x) \neq 0 \quad \text{for } x > \bar{x}.$$

(I.e., $\bar{x}$ is the "rightmost" zero of $f$ on I.)

Since $f$ is continuous on $X$ (Assumption F), the "partial function" $f \mid I$ is continuous in the relative topology of $I$.* The latter topology, by Assumption A, is the ordinary Euclidean topology of the segment. I.e., the "open intervals" $\left\{x \in I: \ x = \theta_x x_* + (1 - \theta_x) x_1, \ 0 \leq \theta' < \theta_x < \theta'' \leq 1\right\}$ are open, etc. Hence we may consider $f$ as a function of the real variable $\theta_x$ on $\theta_x: \left\{0 \leq \theta_x \leq 1\right\}$, with concavity and continuity preserved. We shall denote this function by $f'$, so that $f'(\theta_x) = f(x)$ for $x = \theta_x x_* + (1 - \theta_x) x_1$. I.e., $f'$ is a continuous concave function of the real variable $\theta_x$ defined on the closed segment $[0,1]$. It is known** that the zeros of $f'$ form a closed subset $C$ of $[0,1]$. Hence the least upper bound*** $\bar{\theta}$ of $C$ must be in $C$. Clearly

$$\bar{x} = \bar{\theta}_{x_*} + (1 - \bar{\theta}) x_1$$

satisfies (27)

d.

(28) $\qquad g(\bar{x}) \not> 0 \quad$ (i.e., $g(\bar{x})$ is not an interior point of $P_z$).

For suppose $g(\bar{x}) > 0$. Then, because of the continuity of the "partial function" $g \mid I$, there exists $x_2$, $\bar{x} < x_2 \leq x_1$, such that $g(x_2) > 0$.

[For if $g(\bar{x})$ is in the interior of $P_z$, there exists a neighborhood $N_{g(\bar{x})}$ of $g(\bar{x})$ such that $N_{g(\bar{x})} \subseteq P_z$. Since $g|I$ is continuous, there exists**** a neighborhood $G_{\bar{x}} \subseteq I$ of $\bar{x}$ such that $g(G_{\bar{x}}) \subseteq N_{g(\bar{x})} \subseteq P_z$.

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* Cf. Lefschetz, [25], p. 10, (11.4).

** Rudin, [28], p. 75, No. 3.

*** Its existence is guaranteed by 1.36, p. 11, Rudin, [28].

**** See Kuratowski, [23], p. 73.

I.e., for each $x \in G_{\bar{x}}$, $g(x) \in P_z$. Now $G_{\bar{x}}$ must contain a subinterval of $I$, open in the relative topology of $I$, say $(a, b)$ where $x_* \leq a < \bar{x} < b \leq x_1$. Then $x_2$ can be taken as some point $\bar{x} < x_2 < b$ in the open interval $(\bar{x}, b)$].

But if $g(x_2) > 0$, then, by (14), $f(x_2) \leqq 0$. It follows that $f(x_2) < 0$, since $f(x_2) = 0$ would contradict (27.2).

Thus we have $f(x_2) < 0$, as just shown, and $f(x_1) > 0$ by (21). Consider again the function $f'(\theta_x)$ on $\{\theta : 0 \leq \theta \leq \theta_2\}$ where $x_2 = \theta_2 x_* + (1 - \theta_2) x_1$. Since $f'(\theta_2) < 0$ and $f'(0) > 0$, we must have* $f'(\theta) = 0$ for some $0 < \theta < \theta_2$, i.e., $f(x) = 0$ for some $x_2 < x$. This, however, contradicts (27.2), since $x_2 > \bar{x}$.

e. Thus, by (28), $g(\bar{x})$ is either outside $P_z$ or on its boundary. Since the space $\mathfrak{Z}$ is linear normed, we may use Theorems 1.3, p. 16, Krein and Rutman, [20] to assert the existence of a $z_0^*$ such that

(29) $$0_z^* \neq z_0^* \geqq 0_z^*$$

and

(30') $$z_0^* \, [g(\bar{x})] \leqq 0.$$

Write

(31) $$h(x) \equiv z_0^* \, [g(x)]$$

so that

(30") $$h(\bar{x}) \leqq 0.$$

It is easily seen that $h(x)$ is a concave function of $x$. For, since $g$ is concave and $z_0^* \geqq 0$,

(32) $$z_0^* \, [g(\theta x' + (1 - \theta) x")] \geqq z_0^* \, [\theta g(x') + (1 - \theta) g(x")].$$

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* By the Bolzano-Weierstrass Theorem, cf. Rudin, [28], p. 71, 4.19.

By additivity and homogeneity of $z_o^*$ this becomes

(33) $$z_o^* [g(\theta x' + (1 - \theta) x''] \geqq \theta z_o^* [g(x')] + (1-\theta) z_o^* [g(x'')],$$

i.e.,

(34) $$h (\theta x' + (1-\theta) x'') \geqq \theta h(x') + (1-\theta) h(x'').$$

f. We shall now show that

(35) $$h (x_1) < 0.$$

Let $\bar{\theta}$ be the unique real number satisfying

(36) $$\bar{x} = \bar{\theta} x_* + (1 - \bar{\theta}) x_1.$$

Clearly

(37) $$0 < \bar{\theta} \leq 1.$$

Note that

(38) $$h (x_*) > 0,$$

since $g(x_*) \in \text{Int } P_z$ and then Lemma 1.2, p. 13 in Krein and Rutman, [20] yields $z_o^* [g(x_*)] > 0$. It follows that

(39) $$\bar{\theta} < 1,$$

i.e.,

(39") $$\bar{x} > x_*.$$

(For if $\bar{x} = x_*$, then (38) implies $h(\bar{x}) > 0$ which contradicts (30").)

Now, since $h(x)$ has been shown to be concave,

(40) $$\bar{\theta} h (x_*) + (1-\bar{\theta}) h(x_1) \leqq h(\bar{x}),$$

i.e.,

(41) $$\bar{\theta} h(x_*) - h(\bar{x}) \leqq (1 - \bar{\theta}) h(x_1).$$

By (37), (38), and (30"), the left member of (41) is positive; hence so is the right member, and (because $1-\bar{\theta} > 0$), (35) follows.

g. From the concavity of $h(x)$ and (30") we have

(42) $$\bar{\theta}\, h(x_*) + (1-\bar{\theta})\, h(x_1) = h(\bar{x}) \leqq 0,$$

i.e.,

(43) $$\bar{\theta}\, h(x_*) + (1-\bar{\theta})\, h(x_1) + \epsilon = 0$$

for some

(44) $$\epsilon \geqq 0,$$

so that

(45.1) $$\bar{\theta}\, [h(x_*) - h(x_1)] = -\epsilon - h(x_1)$$

and

(45.2) $$(1-\bar{\theta})\, [h(x_*) - h(x_1)] = \epsilon + h(x_*).$$

Note that $h(x_*) > 0$ by (38) and $h(x_1) < 0$ by (35), hence

(46) $$h(x_*) - h(x_1) > 0.$$

Now, since $f(x)$ is concave and because of (27.1),

(47) $$\bar{\theta}\, f(x_*) + (1-\bar{\theta})\, f(x_1) \leqq f(\bar{x}) = 0.$$

since the inequality in (47) is not affected by the multiplication by the (see (44)) positive number $h(x_*) - h(x_1)$, we obtain from (47), using (45),

(48) $$[-\epsilon - h(x_1)]\, f(x_*) + [\epsilon + h(x_*)]\, f(x_1) \leqq 0$$

which may be rewritten as

(49) $$h(x_*)\, f(x_1) - h(x_1)\, f(x_*) + \epsilon\, [f(x_1) - f(x_*)] \leqq 0.$$

But

(50) $$\epsilon\,[f(x_1) - f(x_*)] \geqq 0$$

in virtue of (44), (21), and (24).

Using (50) in (49) we obtain

(51) $$h(x_*)\,f(x_1) - h(x_1)\,f(x_*) \leqq 0$$

which is the assertion of Lemma 1' if $z^*_{x_{1,1}}$ is taken as $z^*_0$.

For later use we may note that (24) may be strengthened to

(52) $$f(x_*) < 0.$$

(If $f(x_*) = 0$, (51) becomes $h(x_*)\,f(x_1) \leqq 0$ which contradicts (38) and (21).)

3.4 <u>Proof of Lemma 1</u>.

Suppose (21) holds and let $z^*_{x_{1,1}}$ be the functional satisfying (22), (23) whose existence was demonstrated in Lemma 1'.

Because of (38), we may define a $z^*_{x_1}$ by

(53) $$z^*_{x_1}(z) = \left( - \frac{f(x_*)}{z^*_{x_{1,1}}[g(x_*)]} \right) z^*_{x_{1,1}}(z) \quad \text{for all} \quad z \in \mathfrak{Z} .$$

In view of (52) and (38), $z^*_{x_1}(z)$ if and only if $z^*_{x_{1,1}}(z) \geqq 0$, i.e., $z^*_{x_1} \geqq \bar{0}$. We have

(54) $$f(x_1) + z^*_{x_1}[g(x_1)] = f(x_1) - \frac{f(x_*)}{z^*_{x_{1,1}}[g(x_*)]}\, z^*_{x_{1,1}}[g(x_1)]$$

$$= \frac{1}{z^*_{x_{1,1}}[g(x_*)]} \left\{ f(x_1)\, z^*_{x_{1,1}}[g(x_*)] - f(x_*)\, z^*_{x_{1,1}}[g(x_1)] \right\}$$

$$\leqq 0$$

by Lemma 1, so that $z^*_{x_1}$ satisfies 20. Let $g(x_*)$ belong to $P_z$ with a closed sphere $S(g(x_*), \rho_0)$, $\rho_0 > 0$ ($S(a,\sigma)$ denotes the closed sphere with a center at $a$ and radius $\sigma$); since $g(x_*) > 0$, such a sphere must exist.

Also, without loss of generality, we may assume that

$$(55) \qquad || z^*_{x_{1,1}} || = 1.$$

(Since $z^*_{x_{1,1}} [g(x_*)] > 0$, $||z^*_{x_{1,1}}|| \neq 0$. If $z^*_{x_{1,1}}$ satisfies the assertion of Lemma 1', so does $\mu z^*_{x_{1,1}}$ for any $\mu > 0$. Hence (55) can be satisfied with a suitably selected $\mu$).

We have, by Lemma 1.2, p. 13, Krein and Rutman, [20],

$$(56) \qquad z^*_{x_{1,1}} [g(x_*)] \geqq \rho_0 || z^*_{x_{1,1}} || = \rho_0$$

and

$$(57) \qquad ||z^*_{x_1}|| = - \frac{f(x_*)}{z^*_{x_{1,1}}[g(x_*)]} ||z^*_{x_{1,1}}|| = - \frac{f(x_*)}{z^*_{x_{1,1}}[g(x_*)]} \leqq - \frac{f(x_*)}{\rho_0}$$

Now, setting

$$(58) \qquad \rho = - \frac{f(x_*)}{\rho_0} ,$$

we have

$$(59) \qquad ||z^*_{x_1}|| \leqq \rho \quad \text{for all } x_1.$$

This completes the proof of Lemma 1.

4. In order to complete the proof of the main theorem we shall need the following result which is a generalization of Theorem 2 in Karlin and

Bohnenblust ([19.1], p. 156) and may be of independent interest.

Lemma 3.

Let $W$ be a linear normed space and $W^*$ its adjoint space (i.e., $W^*$ is the space of all linear continuous functionals). Let the set $B^* \subseteq W^*$ be bounded and regularly convex while $K$ is a convex cone in $W$.

Furthermore, suppose that for every $w \in K$ there is a $w^*_w \in B^*$ such that

(60) $$w^*_w(w) \geqq 0.$$

Then there exists a $w^*_o \in B^*$ such that

(61) $$w^*_o(w) \geqq 0 \quad \text{for all} \quad w \in K.^{(*)}$$

Proof.

Klee ([18], p. 465, Theorem (12.11) ), generalizing Theorem 7 of Krein and Šmulian ([21]) has shown that if $W$ is linear normed, $A^*_1 \subseteq W^*$, $A^*_2 \subseteq W^*$, $A^*_1$ and $A^*_2$ regularly convex and at least one of $A^*_1$, $A^*_2$ regularly convex, then the set $A^*_1 + A^*_2 \equiv \left\{w^*_1 + w^*_2 : \; w^*_1 \in A^*_1, w^*_2 \in A^*_2\right\}$ is also regularly convex. Clearly, $A^*_1 - A^*_2$ would also be regularly convex, since if $A^*_2$ is regularly convex then so is $-A^*_2 \equiv \left\{ w^*: -w^* \in A^*_2 \right\}$.

One can then show, using the methods of Krein and Šmulian ([21], p. 564, proof of Theorem 4) that if $C^*_1$ and $C^*_2$ is bounded, then there exists a $w_o \in W$ such that

(62) $$\sup_{w^* \in C^*_1} w^*(w_o) < \inf_{w^* \in C^*_2} w^*(w_o).$$

[For the set $C^* = C^*_1 - C^*_2$ is regularly convex and (since $C^*_1 \cap C^*_2 = \Lambda$ ) $0^*_w \notin C^*_1$ so that, for some $w_o \in W$ (by def. of regular convexity)

(*) Theorem 2 of Karlin and Bohnenblust covers the special case where $W$ is Banach and $K$ is closed.

$$\sup_{w^* \in C^*} w^*(w_o) < 0^*_W(w_o) = 0$$

which leads to (62).] By (62), there exists a real number $\alpha$ such that*

(63.1) $\qquad w^*(w_o) < \alpha \qquad$ for all $w^* \in C^*$,

and

(63.2) $\qquad w^*(w_o) > \alpha \qquad$ for all $w^* \in C_2^*$.

[E.g., take $\alpha = \frac{1}{2}\left\{\sup_{w^* \in C_1^*} w^*(w_o) + \inf_{w^* \in C_2^*} w^*(w_o)\right\}$. ]

Now the assertion of Lemma 3 can be rephrased as

(64) $\qquad B^* \cap K^{\oplus} \neq \Lambda$

where $K^{\oplus}$ denotes the conjugate cone of K. If (64) is supposed false, then $w_o$ and $\alpha$ specified in (63) will exist, since both $B^*$ and $K^{\oplus}$ are regularly convex**, disjoint (by denial of (64)) and $B^*$ is bounded (by hypothesis). I.e., for some $w_o \in W$ and real $\alpha$,

(65.1) $\qquad w^*(w_o) < \alpha \qquad$ for all $w^* \in B^*$

and

(65.2) $\qquad w^*(w_o) > \alpha \qquad$ for all $w^* \in K^{\oplus}$.

Since $0^*_W \in K^{\oplus}$, (65.2) yields

(66) $\qquad \alpha < 0$.

On the other hand,

(67) $\qquad w^*(w_o) \geqq 0 \quad$ for all $w^* \in K^{\oplus}$.

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* This statement corresponds to Theorem 1, p. 156, of Bohnenblust and Karlin, [19.1].

** $K^{\oplus}$ has been shown to be regularly convex in PGS, Lemma V.1. Cf. also [20], p. 38. $B^*$ is regularly convex by hypothesis.

[For suppose $w^*(w_o) = \beta < 0$ for some $w^* \in K^+$. Then $w_1^* = 2\frac{\alpha}{\beta} w^*$ is also in $K^{\oplus}$ and $w_1^*(w_o) = 2\frac{\alpha}{\beta} w^*(w_o) = 2\frac{\alpha}{\beta}\beta = 2\alpha < \alpha < 0$ which contradicts (65.2).]

Now, by Cov. 1.3, p. 16, Krein and Rutman, [20], (67) implies

(68) $$w_o \in \bar{K}$$

where $\bar{K}$ denotes the closure (in the norm topology) of K.

Then $w_o$ is either in K or in K'. (K' denotes the derived set of K; its elements are called accumulation points of K.)

Suppose first that

(69) $$w_o \in K.$$

Then, by hypothesis (cf. (60)), there exists a $w^*_{w_o} \in B^*$ such that

(70) $$w^*_{w_o}(w_o) \geqq 0$$

which contradicts (65.1).* Hence suppose that

(71) $$w_o \in K'.$$

I.e., there exists a sequence

(72) $$w = (w_1, w_2, \ldots.) \qquad (w_o \neq w_n \in K)$$

(strongly) converging to $w_o$; i.e., if $\eta > 0$, there exists $n_\eta < \infty$ such that

(73) $$||w_n - w_o|| < \eta \qquad \text{for all } n > n_\eta.$$

By hypothesis (cf. (60)), since $w_n \in K$, we have

---

* In the Karlin and Bohnenblust formulation, this completes the proof, since they assume $K = \bar{K}$.

(74) $w_n^*(w_n) \geqq 0 \qquad n = 1,2,\ldots$

for some $w_n^* \in B^*$.

Now $B^*$ is bicompact (= "compact" in some of the more recent usage) in the weak*-topology.* Furthermore, a bicompact set is compact (= "compact" in some of the recent usage) i.e., if $S$ is an infinite subset of the bicompact set $A$, $S$ has an accumulation point in $A$ (in the weak*-topology).

Hence the set $\Lambda^*$ of the elements of the sequence $w$

(75) $$\Lambda^* = \{ w_1^*, w_2^*, \ldots \}$$

has a point of accumulation, say $w_0^*$ with

(76) $$w_0^* \in B^*.$$

Now, since $B^*$ is bounded, there exists $0 < \rho < \infty$ such that

(77) $$||w^*|| \leqq \rho \qquad \text{for all } w^* \in B^*.$$

choose

(78) $$\eta_0 = -\frac{\alpha}{\rho} > 0$$

There the inequality follows from (66). Then, by (73), there exists $n_{\eta_0} < \infty$ such that

---

* Regular convexity is equivalent to convexity with weak*-closure (cf. Bourgin, (7), p. 655, Theorem 18). Hence $B^*$ is weak*-closed. Being bounded, it is (by def.) a subset of a closed sphere in $w^*$ with some finite radius. Such a sphere is known to be bicompact; cf. Bourgin, (7), p. 656, Theorem 22, Krein and Rutman, [20], p. 39, and Alaoglu, [0,1], Theorem 1:3, p. 255. (The latter two proofs are valid for a normed linear space even if it is not Banach. Bourgin's theorem applies since a linear normed space is locally bounded.) Finally, a weak*-closed subset of a set bicompact in the weak*-topology is itself bicompact (see Lefschetz, [25], p. 18, (23.1)).

(79) $$||w_n - w_o|| < \eta_o \qquad \text{for } n > n_{\eta_o}.$$

Let*

(80) $$\Lambda_o^* = \Lambda^* \sim \left\{w_1^*, w_2^*, \ldots, w_{n_{\eta_o}}^*\right\}$$

Clearly, $w_o^*$ is also an accumulation point of $\Lambda_o^*$.** Now consider the neighborhood $U_o$ of $w_o^*$ in the weak*-topology given by

(81) $$U_o = \left\{ w^*: |w^*(w_o) - w_o^*(w_o)| < \frac{\alpha}{2} \right\}.$$

Then, by definition of a point of accumulation, $U_o$ will contain an element of $\Lambda_o^*$ other than $w_o^*$. Let $w_N^*$ be such an element. I.e.,

(82) $$N > n_{\eta_o}$$

(by definition of $\Lambda_o^*$) and

(83.1) $$w_N^* \in U_o,$$

(83.2) $$w_N^* \in \Lambda_o^*.$$

Now, since

(84) $$\Lambda_o^* \subseteq \Lambda^* \subseteq B^*,$$

we have, in virtue of (65.1) and (83.2),

(85) $$w_N^*(w_o) < \alpha.$$

---

* $A \sim B$ is the set of all elements which are in $A$ but not in $B$ (the set-theoretic difference).

** Kuratowski, [23], Ch. I, § 9, II.9, p. 45.

On the other hand, from (81) and (83.1) we have

(86) $$| w^*_N (w_o) - w^*_o (w_o) | < - \frac{\alpha}{2}$$

Also,

(87) $$| w^*_N (v_N - v_o) | < - \alpha.$$

[For $| w^* (v_n - v_o) | \leqq ||w^*|| . ||v_n - v_o||$. If $w^*$ $B^*$, we have from (77) $|w^* (v_n - v_o) |$ $\rho$ $. || v_n - v_o||$. Now if $n > n_{\eta_o}$, it follows by (73) that $||v_n - v_o|| < \eta_o = - \frac{\alpha}{\rho}$, hence $|w^* (v_n - v_o) | \leqq \rho(- \frac{\alpha}{\rho}) = - \alpha$. Finally, because of (82) and (83.2) with (84), (87) results.]

Now suppose that

(88) $$w^*_N (v_N) \geqq 0.$$

Write

(89) $$w^*_N (v_N) = \epsilon , \epsilon \geqq 0$$

and

(90) $$w^*_N (v_o) = - \beta - \sigma$$

where

(91.1) $$\beta = - \alpha > 0$$

and

(91.2) $$\sigma > 0$$

by (85).

Then

(92) $$w^*_N (v_N - v_o) = w^*_N (v_N) - w^*_N (v_o) = \epsilon - (- \beta - \sigma) = \epsilon + \beta + \sigma.$$

Since $\beta > 0$, $\sigma > 0$ and $\epsilon \geqq 0$,

(93) $$\left| w_N^* (w_N - w_o) \right| = \left| \epsilon + \beta + \sigma \right| = \epsilon + \beta + \sigma > \beta = - \alpha$$

which contradicts (87). Hence either (64) is true, in which case the proof is completed, or (88) is false, i.e.,

(94) $$w_N^* (w_N) < 0$$

which contradicts (74). This completes the proof.

5.1 In virtue of Lemma 1, there exists $0 < \rho < \infty$ such that, for each $x \in X$, there exists a $z_x^*$ such that

(20) $$f(x) + z_x^* [g(x)] \leqq 0, \quad z_x^* \geqq 0, \quad \| z_x^* \| \leqq \rho.$$

Consider now the product space*

$$\mathcal{V} = \mathcal{Y} \times \mathcal{Z}$$

where, by Assumption B, $\mathcal{Y}$ is the set of reals and write**

(95) $$v \equiv (y,z) = (f(x), g(x)) \equiv h(x).$$

Define also $v' \geqq v''$ to mean $v' - v'' \in P_v$ where

(96) $$P_v = \left\{ v: \; v = (y,z), \; y \geqq 0, \; z \geqq 0 \right\}.$$

It is seen that $h(x)$ is concave since both $f$ and $g$ are.

Write

---

* Which is linear normed, see Banach, [4], pp. 181-2.

** The functional symbol in (95) and subsequently is unrelated to $h(x)$ as used in earlier sections.

(97) $$A^* = \left\{ v^* :\ v^* = (y^*, z^*),\ y^* = 1,\ z^* \in S^*_\rho \cap P^\oplus_z \right\}$$

where $y^* = 1$ means that $y^*(y) = y$ for all $y$, and $S^*_\rho$ is the closed sphere of redius $\rho$ in $\mathfrak{Z}^*$ while $P^\oplus_z$ is the conjugate cone of $P_z$, i.e., the set of all $z^* \geqq 0$.

(98) $$v^*_x(v) \leqq 0, \quad v^*_x \in A^*$$

for any

(99) $$v \in h(X)$$

To make the theorem proved in section 4 applicable, we shall show that $A^*$ is regularly convex (it is clearly bounded) and also that a $v^*_x$ satisfying (98) exists for any $v$ in a convex cone containing $h(X)$.

5.2 <u>The regular convexity of $A^*$</u>

As the intersections of two regularly convex sets*, $S^*_\rho \cap P^\oplus_z$ is regularly convex. Now let $v^*_o = (y^*_o, z^*_o) \notin A^*$. To establish the regular convexity of $A^*$ we must show that there exists a $v_o = (y_o, z_o)$ such that

(100) $$\sup_{v^* \in A^*} v^*(v_o) < v^*_o(v_o),$$

i.e.,

(101) $$\sup (c^* \cdot y_o + z^*(z_o)) < c^*_o \cdot y_o + z^*_o(z_o)$$

where $c^*$ and $c^*_o$ are real numbers, $c^* = 1$, and $z^* \in S^*_\rho \cap P^\oplus_z$.

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* That $P^\oplus_z$ is regularly convex is shown in PGS, Lemma (V).1 see also [20], p. 38. The regular convexity of $S^*_\rho$ is well known. Suppose $v^*_o \notin S^*_\rho$. Then there exists a $v_o, ||v_o|| \leqq 1$, such that $|v^*_o(v_o)| > \rho$. If $v^*_o(v_o) > \rho$, then $\sup_{v^* \in S^*_\rho} v^*(v_o) \leqq \rho < v^*_o(v_o)$ so that $S^*_\rho$ is regularly convex. If $v^*_o(v_o) < -\rho$, use $v_1 \equiv -v_o$.

Then we have to find $(y_o, z_o)$ such that

$$(102) \qquad y_o + \sup_{z^* \in S^*_\rho \cap P_z^\oplus} z^*(z_o) < c^*_o \cdot y_o + z^*_o(z_o).$$

Now since $v^*_o \notin A^*$, it must be that $c^*_o \neq 1$ or $z^*_o \notin S^*_\rho \cap P^+_z$. If $z^*_o \notin S^*_\rho \cap P^\oplus_z$, we choose $y_o = 0$ and (102) will be satisfied by some $z_o$ because $S^*_\rho \cap P^\oplus_z$ is regularly convex. If $c^*_o \neq 1$, we take $z_o = 0_z$, and set $y_o > 0$ if $c^*_o > 1$ or $y_o < 0$ if $c^*_o < 1$.

5.3 Let

$$(103) \qquad K = \left\{ v: v = \lambda v_o, \lambda \geqq 0, \; v_o \in \text{conv } h(X) \right\},$$

where, for $S \subseteq V$ we define

$$(104) \qquad \text{conv } S = \left\{ v: \; v = \sum_{i=1}^{m} \lambda_i v_i, \lambda_i \geqq 0, v_i \in S, \sum_{i=1}^{m} \lambda_i = 1, \quad m = 1, \ldots\ldots \right\}$$

(I.e., conv $S$ is the convex hull of $S$.) Then $K$ is a convex cone and, given any $v \in K$, there exists a $v^*_v \in A^*$ such that (98) holds.

<u>Proof</u>.

Let $v \in K$, $\lambda \geqq 0$; then (by (103)) $\lambda v \in K$.

Suppose $v' \in K$, $v'' \in K$. Then

$$(105) \qquad v' = \sum_{i=1}^{m'} \lambda'_i v'_i, \quad v'' = \sum_{i=1}^{m''} \lambda''_i v''_i, \quad v'_i, v''_i \in h(X),$$

and

$$(106) \qquad v' + v'' = \sum_{i=1}^{m} \mu_i v_i, \quad \mu_i \geqq 0, \sum_{i=1}^{m} \mu_i > 0, v_i \in h(X).$$

Hence

(107) $$v' + v'' = \mu \sum_{j=1}^{m} v_j v_j \quad , \quad v_j \in h(X),\ v_j \geqq 0,\ \sum_{j=1}^{m} v_j = 1$$

and $\mu = \sum_{j=1}^{m} \mu_j$. Hence $v' + v'' \in K$ and $K$ has been shown to be a convex cone. Now let $v \in K$. We wish to show that there exists a $v^*_v \in A^*$ such that (98) holds.

We have, by definition of $K$,

(108) $$v = \lambda_o \sum_{i=1}^{m} \lambda_i v_i \quad , \quad v_i = h(x_i),\ x_i \in X,\ \lambda_o \geqq 0,$$
$$\sum_{i=1}^{m} \lambda_i = 1,\ \lambda_i \geqq 0.$$

If $\lambda_o = 0$, then $v = 0_v$ and any element of $A^*$ will satisfy (98). Hence we shall henceforth assume

(109) $$\lambda_o > 0.$$

Consider now

(110) $$x = \sum_{i=1}^{m} \lambda_i x_i.$$

Clearly $x \in X$ since $X$ is assumed convex. But then the concavity of $h$ yields*

(111) $$v' = \sum_{i=1}^{m} \lambda_i \leqq h\left(\sum_{i=1}^{m} \lambda_i x_i\right) \equiv v''$$

Now since $v'' \in h(X)$, there exists a $v^*_{v''} \in A^*$ such that

(112) $$v^*_{v''}(v'') \leqq 0.$$

* This is easily established by induction on $m$, bearing in mind the convexity of $X$.

$v^*_{v''}$ being non-negative*, we have (because of (111))

(113) $$v^*_{v''}(v'' - v') \geqq 0,$$

i.e.,

(114) $$v^*_{v''}(v') \leqq v^*_{v''}(v''),$$

which, in virtue of (112), implies

(115) $$v^*_{v''}(v') \leqq 0,$$

hence, since $\lambda_o > 0$,

(116) $$v^*_{v''}(v) = v^*_{v''}(\lambda_o v') = \lambda_o v^*_{v''}(v') \leqq 0$$

which is the desired result.

5.4 We have now established the fact that for any $v$ in the convex cone $K$ defined is (103) there exists a $v^*_v \in A^*$ such that (98) holds. We have also shown that $A^*$ is regularly convex. Since $A^*$ is clearly bounded, the theorem established in section 4 applies. Hence there exists a $v^*_o \in A^*$ such that

(117) $$v^*_o(v) \leqq 0 \qquad \text{for all } v \in K,$$

hence, in particular,

(118) $$v^*_o(v) \leqq 0 \qquad \text{for all } v \in h(X),$$

i.e., for any $x \in X$, there exists a $z^*_o \geqq 0$, $||z^*_o|| < \rho$, such that

* I.e., if $v^* \in A^*$ and $v \geqq 0$, $v^*(v) \geqq 0$. For if $v(y,z)$, $y \geqq 0$, $z \geqq 0$, $v^* \in A^*$, we have $v^*(v) = y + z^*(z)$ with $z^* \geqq 0$.

(119) $$f(x) + z^*_0 [g(x)] \leqq 0$$

which establishes proposition (T"') in 3.2 above. Thus the proof of the theorem is completed.

## References

[For references other than those listed here see "Programming in General Spaces" (PGS), CCDP Economics No. 2109, by the author of the present paper.]

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