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Application of Safety First to Hedging

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1. Introduction

In this paper we shall find under what conditions short hedging reduces risk and when short hedging is more profitable than long hedging for a merchant following the safety first principle. By safety first we mean that the merchant (who is the hedger) chooses his actions such that he maximizes his expected net income subject to the restriction that the probability that his actual income falls below a disaster level, R, shall not exceed α. The random variables throughout the discussion are prices. We assume that the merchant can buy or sell unlimited quantities of the commodity without affecting the current market price. A short hedge can be described as the purchase of a commodity for immediate delivery accompanied by the sale of an equal quantity of futures contracts. This is the initial pair of transactions in a short hedge. The short hedge is lifted by buying the futures at the same time that the commodity is sold in the spot market. A long hedge occurs when the merchant sells the commodity to someone for delivery later on (a forward sale) and simultaneously buys futures. The long hedge is lifted when the futures are sold and the commodity purchased spot. [2]

We shall need the following notation:

S = unhedged stocks, $S \geq 0$

H = Hedging, $\begin{cases} H > 0 & \text{short hedging} \\ H < 0 & \text{long hedging} \end{cases}$

B = basis now,* B' = basis later, σ_B^2 = Variance B'.

$B < 0$, $B' < 0$, basis = future price less spot price

P = cash price now, P' = cash price later, σ_P^2 = Variance P'

$P > 0$, $P' > 0$

Y = net income. $\sigma_{BP} = 0$ = Covariance of B', P'.

2. Short Hedging

The income of the merchant is given by

$$(1) \qquad Y = H(B - B') + S(P' - P).$$

This is a random variable since it depends on two random variables B' and P'. Its expectation is given by

$$(2) \qquad EY = H(B - \bar{B}') + S(\bar{P}' - P)$$

The merchant then chooses H and S in accord with the following

$$(3) \qquad \max_{H,S} EY \qquad \text{subject to } \frac{\text{Var } Y}{[EY-R]^2} \leqq \alpha > 0$$

where $P_r\,[Y \leq R] \leq \alpha$ and R is the "disaster" level of income. Since the merchant is going to be a short hedger, $H \geq 0$. If both $\bar{P}' - P < 0$ and $B - \bar{B}' < 0$, then the best the merchant can do (if he isn't a long hedger) is to let $H = 0$ and $S = 0$. The interesting case occurs when $B - \bar{B}' < 0$

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* For a description of the basis and its meaning see [2] and [5].

and $\bar{P}' - P > 0$. This means that he expects a short hedge to cost $B - B'$ per unit of the commodity and that if he buys the commodity and holds it he will make a profit of $\bar{P}' - P > 0$ per unit less storage costs. Since a short hedge is costly suppose the merchant decides at first to hold his stocks unhedged. Then $H = 0$ and $S > 0$. What is the best S?

$$EY = S(\bar{P}' - P) \quad \text{and} \quad \text{Var } Y = S^2 \sigma_P^2.$$

The effect of the risk restriction is to limit the size of S. We have

(4) $$\frac{S^2 \sigma_P^2}{(S(P'-P)^2-R)^2} \leq \alpha. \text{ which can be rewritten}$$

(5) $$S^2 (\alpha(\bar{P}' - P)^2 - 2 \alpha R S (\bar{P}' - P) + \alpha R^2 \geq 0$$

Clearly in order to maximize EY, S should be chosen as large as possible. In (5) there are two possibilities with respect to the coefficient of S^2, it is either positive or negative. First suppose that

(6.1) $$\alpha (\bar{P}' - P)^2 - \sigma_P^2 \geq 0.$$

If S is chosen large enough we are assured of meeting the risk restriction. Under these circumstances the quantity of stocks the merchant will hold is limited only by his capital. The interpretation of (6.1) is that the merchant is "sure" of the rise in the cash price so that he doesn't have to worry about the risk restriction.

(6.2) $$\alpha (\bar{P}' - P)^2 - \sigma_P^2 < 0.$$

This means that the merchant is not "sure" of the rise in price and if he holds too much stocks he will violate the risk restriction in (5). In (6.2) we see that $(\bar{P}' - P) < \frac{\sigma_P}{\sqrt{\alpha}}$. In this case we note that in (5) the expression is positive for $S = 0$ and negative for S large enough so that there is a positive root S_o which tells us the largest quantity of unhedged stocks that can be held by the merchant without violating the risk restriction.

$$(7) \qquad S_o = \frac{R\,[\alpha(\bar{P}' - P) - \sigma_P\sqrt{\alpha}\,]}{[\,\alpha(\bar{P}' - P)^2 - \sigma_P^2\,]}$$

We shall now argue that the amount of unhedged stocks that can be held without violating the risk restriction can be increased beyond S_o provided H is greater than zero.

We first note that $B - \bar{B}' \geq B$ or $-\bar{B}' \geq 0$ because the basis is always taken to be negative. This is a simplification which may not be true for the individual merchant. However if we assume that marginal storage costs are constant and the same for all merchants then the basis is always negative. It also follows from this that it is always cheaper for a hedger not to deliver on his futures contract but rather to lift his hedge by selling spot and buying futures. The risk restriction in (3) can be now written as

$$(8) \qquad G(H,S) = H^2[\alpha(B - \bar{B}')^2 - \sigma_B^2] + S^2\,[\alpha(\bar{P}' - P)^2 - \sigma_P^2]$$

$$+ 2HS\,[\alpha(\bar{P}' - P)\,(B - \bar{B}') - \sigma_{BP}]$$

$$-2\alpha HR\,(B - \bar{B}') - 2\alpha\,SR\,(\bar{P}' - P) + \alpha R^2 \geq 0$$

For simplicity in notation let $b = B - \bar{B}'$ and $p = \bar{P}' - P > 0$. Then $b < 0$ and $p > 0$.

Along the line $H = 0$ in the HS plane, $G(H,S)$ remains positive until S_o is reached. Beyond S_o, on $H = 0$, G is negative.

Let $\quad \dfrac{S - S_o}{H} = \theta$.

We shall show that along this line in the $H - S$ plane the function $G(H,S)$ is positive (under suitable conditions) so that expected income can be increased. We shall assume $\sigma_{BP} = 0$ to simplify the algebra. However this assumption is not essential.

(9) $\quad G(H, S_o + \theta H) = H^2 (\alpha b^2 - \sigma_B^2) + (S_o + \theta H)^2 (\alpha p^2 - \sigma_P^2)$

$\quad + 2\alpha H(S_o + \theta H) bp - 2\alpha HRb - 2\alpha(S_o + \theta H) Rp + \alpha R^2$

Since $S_o^2(\alpha p^2 - \sigma_P^2) - 2\alpha RS_o p + \alpha R^2 = 0$ we find that (9) can be simplified to

(10) $\quad G(H, S_o + \theta H) = H^2 [(\alpha b^2 - \sigma_B^2) + \theta^2(\alpha p^2 - \sigma_P^2) + 2\alpha\theta bp]$

$\quad + H [2\theta S_o(\alpha p^2 - \sigma_P^2) + 2\alpha S_o bp - 2\alpha Rb - 2\alpha\theta\alpha R_p] \overset{?}{>} 0$

for a suitable choice of θ. Oneeroot is $H = 0$. In (10) the coefficient of H is negative and $G(0, S_o + 0) = 0$. Hence in order that G be positive for some H greater than zero we require that the coefficient of H^2 be positive.

The coefficient of H can be written as

(11) $\quad \theta^2(\alpha p^2 - \sigma_P^2) + 2\alpha bp\theta + (\alpha b^2 - \sigma_B^2) \overset{?}{\geq} 0$

Since $\alpha p^2 - \sigma_P^2) < 0$, this relation will be negative for θ large enough. It will be positive only if it has a positive root θ_o. In order to have a positive root

(12.1) $\quad \alpha b^2 - \sigma_B^2 > 0$.

When (12.1) is satisfied then (11) will be positive in the neighborhood of $\theta = 0$. The interpretation of (12.1) is that the merchant is more sure of the movement of the basis than he is of the movement of the spot price. In order to make expected income as large as possible we would like θ to be as large as possible. That is, we would like to hold as much unhedged stocks relative to short hedged stocks as possible. The largest value of θ is the root of (11). Any θ between zero and the root of (11) will satisfy the risk restriction.

The root is given by

$$(12.2) \qquad \theta = \frac{-\alpha bp - \sqrt{\alpha b^2 \sigma_P^2 + \alpha p^2 \sigma_B^2 - \sigma_B^2 \sigma_P^2}}{(\alpha p^2 - \sigma_P^2)}.$$

Since $p(S - S_o) + bH > 0$ implies that

$p\,\theta + b > 0$ implies

$$(12.3) \qquad \theta > -\frac{b}{p}$$

which is the second condition that must be met for the policy of mixing unhedged and short hedged stocks to be profitable. When we combine (12.2) which tells us the <u>largest value</u> for θ with (12.3) which tells us the smallest value for θ we obtain

$$(12.4) \qquad -\frac{b}{p} < \theta < \frac{-\alpha b p - \sqrt{\alpha b^2 \sigma_P^2 + \alpha p^2 \sigma_B^2 - \sigma_B^2 \sigma_P^2}}{(\alpha p^2 - \sigma_P^2)}$$

This condition together with (12.1) tells us when it will be profitable to hold hedged and unhedged stocks simultaneously. To make expected income as large as possible we must let θ be as large as possible.

If we put (8) into canonical form by completing squares, for $G(H,S) = 0$ we obtain after some calculation

$$(13) \qquad (\alpha p^2 - \sigma_P^2)\, X^2 + \frac{\left(1 - \frac{\alpha b^2}{\sigma_B^2} - \frac{\alpha p^2}{\sigma_P^2}\right)\sigma_B^2\sigma_P^2\, Z^2}{(\alpha P^2 - \sigma_P^2)} + \frac{\alpha R^2}{\left(1 - \frac{\alpha b^2}{\sigma_B^2} - \frac{\alpha p^2}{\sigma_P^2}\right)} = 0$$

which represents a hyperbola in the XZ-plane.

Keeping this in mind let us return to the original $G(H,S)$ in (8).

$$\begin{cases} (14.1) \quad \dfrac{\partial G}{\partial H} = 2H(\alpha b^2 - \sigma_B^2) + 2\alpha b(pS - R) \\[2ex] (14.2) \quad \dfrac{\partial G}{\partial S} = 2S(\alpha p^2 - \sigma_P^2) + 2\alpha p(Hb - R) \end{cases}$$

$$\therefore \quad \frac{dH}{dS} = -\frac{\frac{\partial G}{\partial S}}{\frac{\partial H}{\partial S}} = -\frac{S(\alpha p^2 - \sigma_P^2) + \alpha p(Hb - R)}{H(\alpha b^2 - \sigma_B^2) + \alpha b(pS - R)}$$

$$\left.\frac{dH}{dS}\right|_{(S_o,0)} = -\frac{S_o(\alpha p^2 - \sigma_P^2) - \alpha Rp}{\alpha b(pS - R)} < 0$$

since $pS - R > 0$. and

$$\left.\frac{dH}{dS}\right|_{(0,H)} = -\frac{\alpha p\,(Hb - R)}{H(\alpha b^2 - \sigma_B^2) - \alpha Rb} > 0 \quad \text{since } \alpha b^2 - \sigma_B^2 > 0$$

This gives us the following picture in the HS plane:

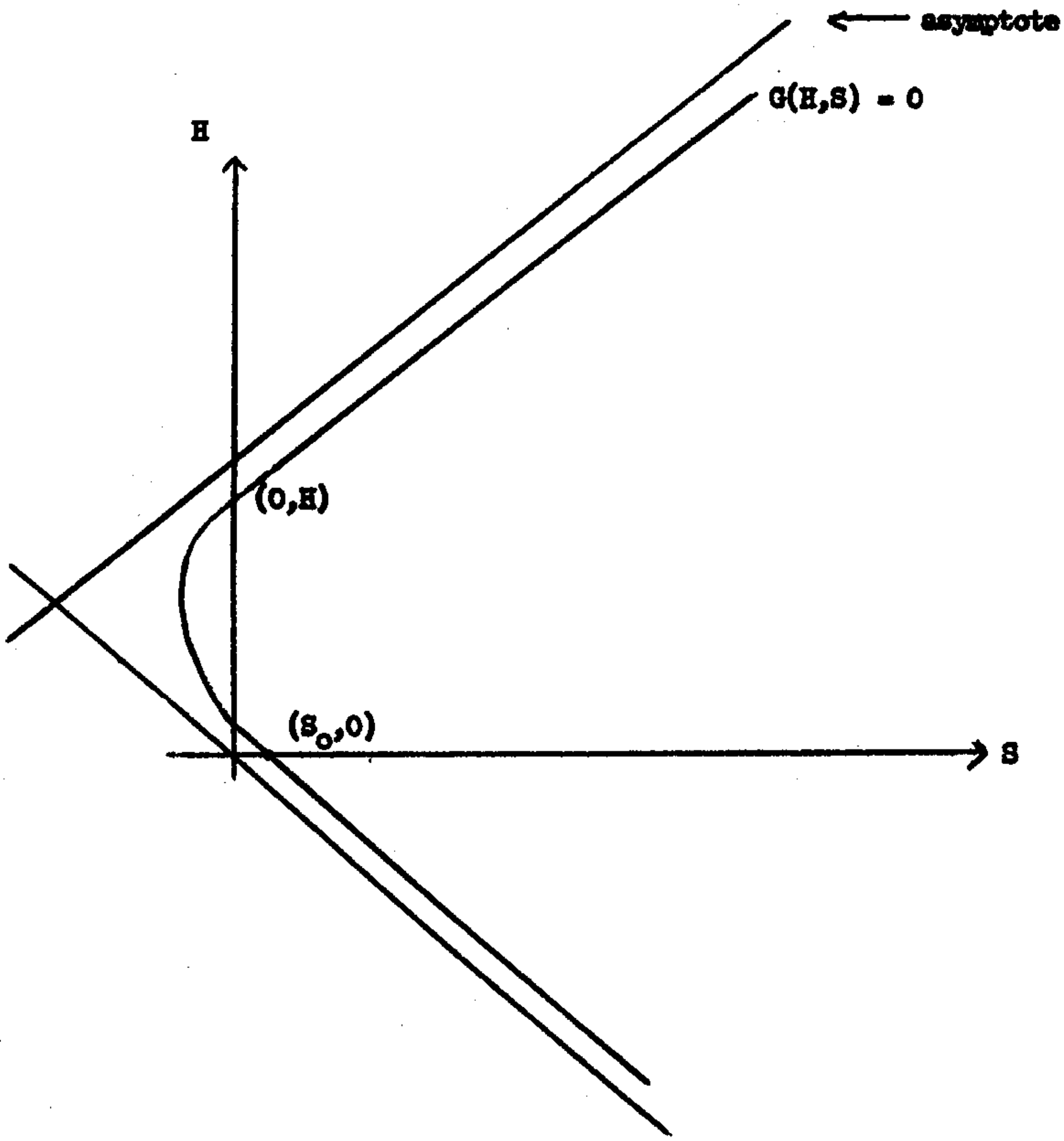

Fig. 1

The slope of the asymptot~~es~~ai of the hyperbola are then

$$\frac{\pm\ \sigma_B\,\sigma_p\,\sqrt{1-\frac{\alpha p^2}{\sigma_p^2}-\frac{\alpha b^2}{\sigma_B^2}}}{(\alpha p^2-\sigma_p^2)} = \theta + \frac{\alpha\, b\, p}{\alpha\, p^2-\sigma_p^2}$$

Hence the upper limit of the ratio between hedged and unhedged stocks must be less than the slope of the asymptote of the hyperbola of the risk restriction.

The conclusion is that by holding S_o stocks unhedged and a θ mixture of hedged and unhedged stocks in addition to S_o then an unlimited total quantity of stocks can be held without violating the risk restriction. The usual limit to the total quantity of stocks will presumably be the capital commanded by the merchant. Even though the merchant loses money on his hedged stocks they serve the purpose of reducing the risk on his unhedged stocks.

3. Long Hedging

Under the conditions previously set forth it would appear that long hedging is expected to be profitable since $B - \bar{B}' < 0$ and by taking $H < 0$ we find that $H(B - \bar{B}') > 0$. The profit per unit of long hedged stocks is $-(B - \bar{B}') = -b$. We shall look into this under the conditions assumed in the previous section:

$$b < 0,\ p > 0,\ \alpha b^2 - \sigma_B^2 > 0, \quad \text{and} \quad \alpha p^2 - \sigma_P^2 < 0.$$

It is easy to see that for $H < 0$, $S = 0$, an infinite income is expected because the coefficient of H^2 in the risk restriction is positive when we let $S = 0$.

Suppose that the amount of "capital" available to the merchant is K. Then the maximum quantity of long hedged stocks he can hold is $-K/B$. Hence his expected income if he is a long hedger is

$$EY_2 = K \cdot \frac{-b}{B}.$$

Now let us compare this with what he expects to get if he holds S_o of his stocks unhedged and thereafter keeps the proportion θ between his unhedged and short hedged stocks. $\theta = \dfrac{S - S_o}{H}$ where $S - S_o = \Delta S$ and $S =$ total unhedged stocks, $H > 0$. The capital available for the short hedge mixture is $K - P\, S_o$. The cost of the mixture is $B - P\,\theta < 0$. Hence the number of units of the mixture he can hold is

(15) $$\frac{K - P\, S_o}{P\,\theta - B} = A.$$

But A which equals the number of units of mixture is also equal to $H + \Delta S$ and $\Delta S = \theta H$.

$$\therefore \quad A = H(\theta + 1),\ \Delta S = \frac{\theta}{\theta + 1}\, A, \quad \text{and} \quad H = \frac{A}{\theta + 1}.$$

The expected income on the mixture is $p.\Delta S + bH$ and this is equal to $\frac{A}{\theta + 1}(p\theta + b)$. Hence the expected income from a mixture of short hedged and unhedged stocks is given by:

(16) $$EY_1 = p\, S_o + \frac{p\theta + b}{\theta + 1}\, A.$$

Our task is to find under what conditions $EY_1 > EY_2$.

(17) $$EY_1 - EY_2 = p\,\frac{[S_o(\theta + 1)\,(P\theta - B) + \theta(K - PS_o)]}{(\theta + 1)\,(P\theta - B)} + b\,\frac{[(K - PS_o)B - K(\theta + 1)\,(P\theta - B)]}{B(\theta + 1)\,(P\theta - B)} \overset{?}{>} 0$$

The coefficients of p and b are both positive and $p > 0$, $b < 0$. Hence from (17) we find that $EY_1 - EY_2 > 0$ if and only if

(18) $$-\frac{b}{p} < \frac{B\,[S_o(\theta + 1)\,(P\theta - B) + \theta(K - PS_o)]}{[(K - PS_o)\,B - K\,(\theta + 1)\,(P\theta - B)]} > 0$$

Since $\theta > -\frac{b}{p}$ (18) will be satisfied when $\theta >$ the expression on the right in (18). From this last statement and (18) we find that

(19) $\quad - BS_o > \theta K$ implies that $EY_1 > EY_2$.

Hence we find that $S_o > \theta \frac{K}{-B}$. The greater is the capital K, the less likely is it that (19) will be met since S_o does not depend on K. We recognize $\frac{K}{-B}$ as the quantity of <u>long</u> hedged stocks. Hence $S_o > \theta$ multiplied by long hedged stocks. Since we know that $\theta > -\frac{b}{p}$ if we let $\theta = -\frac{b}{p}$ we find that (19) becomes $pS_o > -b$ times long hedged stocks. Hence when expected profits on the unhedged stocks S_o exceeds the expected net income from long hedging then surely $EY_1 > EY_2$. However the condition stated in (19) is not necessary and sufficient but is only sufficient since $EY_1 - EY_2 > 0$ when θ is less than the right hand expression in (18). The condition in (18) is necessary and sufficient but doesn't seem to have a simple interpretation.

We can sum up briefly by saying that even though long hedging per se is profitable it might still be the case that short hedging and unhedged stocks permits greater profits in toto than long hedging and this is the meaning of (18).

There are at least two consequences that follow from this theory. First that the gains from short hedging per se should be negative (see [3]). Second we should find that the quantity of short hedging depends not only on the basis but also on the spot price (see [4]).

References

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