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Can We Measure Marginal Utility by Observing Choices Among Uncertain Prospects?

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I. Definitions and notation

An uncertain prospect X^1 is a set of probabilities $p_i^1 \geq 0$ associated with mutually exclusive outcomes x_i^1, $i = 1.....n$, $\sum_i p_i^1 = 1$.

The possible outcomes x_i^1 are called the element of X^1; for convenience, we assume that they are expressed in money-terms, all real goods being reduced to monetary equivalents. This assumption is not essential, but simplifies the analysis.

The sum of money x_i^1 made available at the beginning of period t is written x_{ti}^1. The probability that x_i^1 will be made available in period t is written p_{ti}^1.

We represent X^1 by $(x_1^1 .....x_n^1; p_1^1 .....p_n^1)$ or $(x_i^1; p_i^1)$

Given two independent uncertain prospects X^1 and X^2, the probability distribution of the sum of their elements is represented by $(x_1^1 + x_1^2, ..., x_1^1 + x_n^2, x_2^1 + x_1^2 ....x_2^1 + x_n^2, ..., x_n^1 + x_1^2....x_n^1 + x_n^2 ; p_1^1 p_1^2, ..., p_1^1 p_n^2, p_2^1 p_1^2, ...p_2^1 p_n^2, ..., p_n^1 p_1^2 .....p_n^1 p_n^2)$ or $(x_i^1 + x_j^2 ; p_i^1 p_j^2)$.

The prospect $X^1 = (x^1_{1i} + x^1_{2j} \ldots\ldots + x^1_{s\ell}; p^1_{1i}\, p^1_{2j} \ldots p^1_{s\ell})$ is written $(y^1_k; p^{y^1}_k)$ where y^1_k stands for a particular stream of outcomes and $p^{y^1}_k$ for the probability of occurrence of that particular stream; $i, j, \ldots \ell = 1 \ldots\ldots n$; $k = 1 \ldots\ldots n^s$.

For $p^{y^1}_k = 1$, the present value of $y^1_k = \bar{y}^1_k$.

The uncertain prospect consisting of the probabilities r, $1 - r$ and of the elements X^1, X^2 is represented by $[x^1_i, x^2_j; p^1_i\, r, p^2_j(1 - \ )]$.

The wealth, or net worth, W, of an individual, is defined in terms of the future returns to his assets and abilities. Superscripts are used to distinguish various wealth prospects.

When the future returns are known with certainty, we write w_t for the returns in period t, $\bar{w}_t$ for the present value of w_t, and $W = \sum_{t=1}^{s} \bar{w}_t$.

When the future returns are uncertain, $w_t = (w_{ti}; p^w_{ti})$ and $W = (w_{1i} + w_{2j} \ldots\ldots + w_{s\ell}; p^w_{1i}\, p^w_{2j} \ldots p^w_{s\ell}) = (Z_k; p^z_k)$ when Z_i stands for a particular income stream and p^z_i for the probability of occurrence of that particular income stream. $i, j, \ldots\ldots \ell = 1 \ldots\ldots n$; $k = 1 \ldots\ldots n^s$

For $p^z_k = 1$, the present value of $Z_k = \bar{Z}_k$.

I and P stand for the relations of indifference and preference among prospects.

Given a prospect X^1 and an arbitrary function U, we write $\Sigma p^1_i\, U(x^1_i) = \mu^1_1$, $\Sigma p_i\, [U(x^1_i) - \mu^1_1]^2 = \mu^1_2$, $\Sigma p_i\, [U(x^1_i) - \mu^1_1]^3 = \mu^1_3$.

Additional notation is introduced in Section V.

II. Bernoulli approach

1) There exists a utility function U

2) $U = U(W)$

3) $U = \log W$

4) $X^1 \, I \, X^2$ if and only if $\Sigma p_i^1 \log (W + x_i^1) - \log W = \Sigma p_i^2 \log (W + x_i^2) - \log W$

$X^1 \, P \, X^2$ if and only if $\Sigma p_i^1 \log (W + x_i^1) - \log W > \Sigma p_i^2 \log (W + x_i^2) - \log W$

Comments

1) The ordering of uncertain prospects is invariant under linear transformations of U, but only under such transformations.

2) If $X^1 = (x_i^1 \, ; \, p_i^1)$, $X^2 = (x_i^2 \, ; \, p_i^2)$, $W = (Z_k ; p_k^z)$, then:

$U(W) = \Sigma p_k^z \log \bar{Z}_k$, and

$X^1 \, I \, X^2$ if and only if $\sum_k \sum_i p_k^z p_i^1 \log (\bar{Z}_k + x_i^1) - \Sigma p_k^z \log \bar{Z}_k$

$= \sum_k \sum_i p_k^z p_i^2 \log (\bar{Z}_k + x_i^z) - \Sigma p_k^z \log \bar{Z}_k$

3) If $X^1 = (y_k^1 ; p_k^{y^1})$, $X^2 = (y_k^2 ; p_k^{y^2})$, $W = (Z_k ; p_k^z)$, then:

$X^1 \, I \, X^2$ if and only if

$$\sum_k \sum_j p_k^z p_k^{y^1} \log (\bar{Z}_k + \bar{y}_j^1) - \Sigma p_k^z \log \bar{Z}_k = \sum_k \sum_j p_k^z p_j^{y^2} \log (\bar{Z}_k + \bar{y}_j^2) - \Sigma p_k^z \log \bar{Z}_k$$

4) Such a theory should be tested empirically. If it fails to yield accurate predictions, this may be due to several reasons, and in particular:

$U = U^*(W) \neq \log W$

$U = U(\mu_1^W, \mu_2^W, \mu_3^W)$

III. Marschak approach

Basic postulates:

1) There exists a complete, transitive ordering of uncertain prospects.
2) That ordering is continuous in the probability space
3) X^1 P X^2 for some X^1, X^2 such that some $0 < p_i^1 < 1$, $0 < p_j^2 < 1$.
4) if X^1 I X^2, then, for any X^3

$$[x_i^1, x_j^3 ; p_i^1 r, p_j^3 (1-r)] \text{ I } [x_i^2, x_j^3 ; p_i^2 r, p_j^3 (1-r)]$$

Theorem: There exists an f, defined up to a linear transformation, such that:

$$X^1 \text{ I } X^2 \text{ if and only if } \Sigma p_i^1 f(x_i^1) = \Sigma p_i^2 f(x_i^2)$$

$$X^1 \text{ P } X^2 \text{ if and only if } \Sigma p_i^1 f(x_i^1) > \Sigma p_i^2 f(x_i^2)$$

Comments:

1) This theory could be reformulated as follows:

Given postulates 1-3, the hypothesis that there exists an f such that X^1 I X^2 if and only if $\Sigma p_i^1 f(x_i^1) = \Sigma p_i^2 f(x_i^2)$, if true, implies that f is defined up to a linear transformation, and that postulate 4 holds. Plausibility-arguments in favor of the hypothesis:

-- postulates 1-3 are merely consistency requirements

-- postulate 4, which is equivalent with the hypothesis, seems reasonable.

2) Given postulates 1-3 and $W = \sum_t \bar{w}_t$, make the hypothesis that there exists an f such that

$$X^1 \text{ I } X^2 \text{ if and only if } \Sigma p_i^1 f(W + x_i^1) = \Sigma p_i^2 f(W + x_i^2).$$

If this hypothesis is true, the hypothesis that f is identical with U^*, up to a linear transformation, is not disproved. However, there is no proof of that identity (the proof by Allais is incorrect).

3) If $X^1 = (x_i^1; p_i^1)$, $X^2 = (x_i^2 ; p_i^2)$, $W = (Z_k; p_k^z)$, then, given postulates 1-4, the theorem may become either:

A. There exists an f, defined up to a linear transformation, such that

$X^1 \; I \; X^2$ if and only if $\sum_k \sum_i p_k^z \, p_i^1 \, f(\bar{Z}_k + x_i^1) = \sum_k \sum_i p_k^z \, p_i^2 \, f(\bar{Z}_k + x_i^2)$

or:

B. There exists an f, defined up to a linear transformation, such that

$X^1 \; I \; X^2$ if and only if $\sum p_i^1 \, f(x_i^1) = \sum_k p_i^2 \, f(x_i^2)$, given $W = (Z_k; p_k^z)$

4) If $X^1 = (y_k^1; p_k^{y^1})$, $X^2 = (y_k^2 ; p_k^{y^2})$, $W = (Z_k; p_k^z)$, then, given postulates 1-4, the theorem may become either:

A. There exists an f defined up to a linear transformation, such that

$X^1 \; I \; X^2$ if and only if $\sum_k \sum_j p_k^z \, p_j^{y^1} f(\bar{Z}_k + \bar{y}_j^1)$

$= \sum_k \sum_j p_k^z \, p_j^{y^2} f\,(\bar{Z}_k + \bar{y}_j^2)$ or

B. There exists an f, defined up to a linear transformation, such that

$X^1 \; I \; X^2$ if and only if $\sum p_k^{y^1} \, f(\bar{y}_k^1) = \sum p_k^{y^2} \, f(\bar{y}_k^2)$, given

$W = (Z_k ; p_k^z)$

IV. <u>Allais approach</u>

1) There exists a utility function, U

2) $U = U(W)$, $\frac{\partial U}{\partial W} > 0$, $\frac{\partial^2 U}{\partial W^2} < 0$

[U could be defined up to a linear transformation by means of introspection and psychological observations].

3) There exists an f such that

$X^1 \; I \; X^2$ if and only if $f(\mu_1^1, \mu_2^1) = f(\mu_1^2, \mu_2^2)$

4) $\dfrac{\partial f}{\partial \mu_1} > 0, \quad \dfrac{\partial f}{\partial \mu_2} \gtreqless 0$

5) when $\dfrac{\partial f}{\partial \mu_2} = 0, \quad X^1 \; I \; X^2$ if and only if $\mu_1^1 = \mu_1^2 = \Sigma \, p_i^1 \, U(x_i^1)$

$$= \Sigma \, p_i^2 \, U \, (x_i^2)$$

Comments:

1) Writing W^1 for $(W + X^1)$ and W^2 for $(W + X^2)$, this should be restated:

There exists an f such that

$X^1 \; I \; X^2$ if and only if $f(\mu_1^{W^1}, \mu_2^{W^1}) = f(\mu_1^{W^2}, \mu_2^{W^2})$

$$\frac{\partial f}{\partial \mu_1^W} > 0, \quad \frac{\partial f}{\partial \mu_2^W} \gtreqless 0$$

When $\dfrac{\partial f}{\partial \mu_2^W} = 0$, $X^1 \; I \; X^2$ if and only if

$$\mu_1^{W^1} = \mu_1^{W^2} = \Sigma \, p_i^1 \, U \, (W + x_i^1) = \Sigma \, p_i^2 \, U \, (W + x_i^2)$$

2) When $W = (Z_k ; p_k^z)$,

$$\mu_1^{W^1} = \sum_k \sum_i p_k^z \, p_i^1 \; U \, (\bar{Z}_k + x_i^1) \neq \mu_1^W + \mu_1^1$$

$$\mu_2^{W^1} = \sum_k \sum_i p_k^z \, p_i^1 \, [U(\bar{Z}_k + x_i^1) - \mu_1^{W^1}]^2 \neq \mu_2^W + \mu_2^1$$

3) Why not introduce μ_3 as well as μ_1 and μ_2 into the analysis?

4) if $\frac{\partial f}{\partial \mu_2^W} < 0$, does it follow that: $\frac{\partial f}{\partial \mu_2^1} < 0$? $\frac{\partial f}{\partial \mu_2^{W^1}} < 0$?

V. <u>The problem of present values of future incomes</u>.

Write c_t for the consumption and s_t for the savings of an individual in period t, $t = 1 \ldots\ldots s$.

When $W = \sum_t \bar{w}_t$, $c_t = c_t(W)$

When $W = (Z_k; p_k^z)$, $c_t = [c_{ti}(W); p_{ti}^c]$ $\qquad t = 1 \ldots\ldots s$, $i = 1 \ldots\ldots n^t$

Hypothesis:

a) There exists a V, defined up to a monotonic transformation, and a v, defined up to a linear transformation, such that:

$W^1 \; I \; W^2$ if and only if

$$V\left(\sum_i p_{1i}^{c^1} v[c_{1i}^1(W^1)] \ldots\ldots, \sum_i p_{si}^{c^1} v[c_{si}^1(W^1)]\right)$$

$$= V\left(\sum_i p_{1i}^{c^2} v\,[c_{1i}^2\,(W^2)] \ldots\ldots, \sum_i p_{si}^{c^2} v\,[c_{si}^2\,(W^2)]\right)$$

b) V is homogeneous of some positive degree, different from 1, in

c_{ti}; $\qquad t = 1 \ldots\ldots s$, $i = 1 \ldots\ldots n^t$

The second part of the hypothesis is equivalent to assumption II of Modigliani and Brumberg. It implies that V be homogeneous in v, and v in c_{ti}. Without loss of generality, we may write $v(c_{ti}) = \tau c_{ti}^{\lambda}$, $\lambda > 0$, $\lambda \neq 1$, $\tau > 0$.

Simplified model

$s = 2$; no rate of interest, no carry-over of savings beyond the second period, no uncertainty attached to the needs or to the value of the savings. $W = (Y_1 + Y_{21};\ p_i)$

$$V = \alpha\, c_1^\lambda + \beta \sum p_i\, c_{21}^\lambda + \gamma \sum p_i\, [\beta\, c_{21}^\lambda - \beta\, (r_{21}\, c_1)^\lambda]$$

$$= (\alpha - \gamma\, \beta\, r_{21}^\lambda)\, c_1^\lambda + \beta\, (1 + \gamma) \sum p_i\, c_{21}^\lambda$$

$$= (\alpha - \gamma\, \beta\, r_{21}^\lambda)\, c_1^\lambda + \beta\, (1 + \gamma) \sum p_i\, (Y_1 + Y_{21} - c_1)^\lambda$$

$\gamma \geqq 0$,

r_{21} is the ratio $\dfrac{c_2}{c_1}$ which would be achieved under conditions of certainty. That ratio is fixed as a consequence of our assumption of homogeneity. V is (tentatively!) written in an additive form:

1) to make the degree of its homogeneity in c_1, c_{21} independent of the arbitrary choice of a time-unit without any further assumption regarding the change in λ induced by a change in the length of the time-unit.

2) on the ground that, under conditions of certainty, monotonic transformations on V can account for the interdependence between c_1 and c_2. ¶ The term $\gamma \sum p_i\, [\beta\, c_{21}^\lambda - \beta\, (r_{21}\, c_1)^\lambda]$ corresponds to the idea that an expectation of improvement in the consumption level is a form of present consumption.

Since under conditions of certainty $\dfrac{\partial V}{\partial c_1} = \dfrac{\partial V}{\partial c_2} = \lambda\, \alpha c_1^{\lambda-1} = \lambda\, \beta\, c_2^{\lambda-1}$, $\dfrac{\lambda\alpha}{\lambda\beta} = \left(\dfrac{c_2}{c_1}\right)^{\lambda-1}$ and $\dfrac{\alpha}{\beta} = r_{21}^{\lambda-1}$; set $\alpha = 1$ (choice of unit).

Then $\beta = r_{21}^{1-\lambda}$ and

$$\text{(I)} \quad V = (1 - \gamma r_{21}) c_1^{\lambda} + \beta (1 + \gamma) \Sigma p_i (Y_1 + Y_{2i} - c_1)^{\lambda}$$

Maximizing (I) with respect to c_1, we get:

$$\frac{\partial V}{\partial c_1} = \lambda (1 - \gamma r_{21}) c_1^{\lambda-1} - \lambda \beta (1 + \gamma) \Sigma p_i (Y_1 + Y_{2i} - c_1)^{\lambda-1} = 0$$

$$c_1^{\lambda-1} = \frac{\beta(1 + \gamma)}{1 - \gamma r_{21}} \Sigma p_i (Y_1 + Y_{2i} - c_1)^{\lambda-1}$$

or $$(Y_1 - s_1)^{\lambda-1} = \frac{\beta(1 + \gamma)}{1-\gamma r_{21}} \Sigma p_i (Y_{2i} + s_1)^{\lambda-1}$$

For $\beta = 1$, $\lambda < 1$, $\Sigma p_i Y_{2i} = Y_1$, $\gamma = 0$, this equation will be satisfied only by positive values of s_1^* and $\frac{\partial s_1^*}{\partial \beta} > 0$, $\frac{\partial s_1^*}{\partial \gamma} > 0$.

Extension of the model

A. 3-periods case: $s = 3$; $\gamma = 0$; $W = (Y_1 + Y_{2i} + Y_{3j};\ p_{2i}\ p_{3j})$

$V = \alpha c_1^{\lambda} + \beta \Sigma p_{2i} c_{2i}^{\lambda} + \delta \Sigma p_{3i} c_{3i}^{\lambda}$. If c_1 and y_{2i} are given, we may write:

$$V (c_{2i}, c_3 \mid c_1, y_{2i}) = \beta c_{2i}^{\lambda} + \delta \Sigma p_{3j} [Y_{3j} + Y_{2i} + Y_1 - c_1 - c_{2i}]^{\lambda}$$

For a given c_1

$$V (c_2, c_3 \mid c_1) = \Sigma p_{2i} V (c_{2i}, c_3 \mid c_1, Y_{2i})$$

$$= \beta \Sigma p_{2i} c_{2i}^{\lambda} + \delta \Sigma \Sigma p_{2i} p_{3j} [Y_{3j} + Y_{2i} + Y_1 - c_1 - c_{2i}]^{\lambda},$$

and

$$V(c_1, c_2, c_3) = \alpha c_1^{\lambda} + \beta \Sigma p_{2i} c_{2i}^{\lambda} + \delta \Sigma \Sigma p_{2i} p_{3j} [Y_{3j} + Y_{2i} + y_1 - c_1 - c_{2i}]^{\lambda}$$

The complete formulation ($\gamma \neq 0$) would be:

$$V = \alpha c_1^\lambda + \beta \Sigma p_{2i} c_{2i}^\lambda + \gamma \Sigma p_{2i} (\beta c_{2i}^\lambda - \beta r_{21}^\lambda c_1^\lambda) + \delta \Sigma p_{3i} c_{3i}^\lambda$$

$$+ \epsilon \Sigma p_{3i} (\delta c_{3i}^\lambda - \delta r_{31}^\lambda c_1^\lambda) + \zeta \Sigma \Sigma p_{2i} p_{3j} (\delta c_{3j}^\lambda - \delta r_{32}^\lambda c_{2i}^\lambda)$$

B. Uncertainty about future needs: β becomes $(\beta_i; p_i^\beta)$

Uncertainty about the value of savings: c_{2i} becomes $\Sigma p_{2i} p_j^s (Y_{2i} + a_j s_1)$

In this case, even with $\beta = 1$, $\lambda < 1$, $\Sigma p_{2i} Y_{2i} = Y$, $\gamma = 0$, $\Sigma p_j^s a_j s_1 = s_1$, s_1^* may well be negative.

Problem of allocation of savings between purchases of durable goods and other forms of investment: write $\bar{S}_1$ for the savings invested in the form of durable, S_1 for the savings invested in other forms, and $a_i Y_1$ for Y_{2i}. It seems reasonable to assume then that $c_{2i} = a_i(Y_1 + \bar{S}_1) + S_1$ -- or eventually

$$c_{2i} = a_i(Y_1 + b_i \bar{S}_1) + S_1$$

A special treatment of savings in the form of insurance policy should also be considered.

VI. Conclusions

Only the Marschak approach provides a basis for systematic measurement of the marginal utility of wealth.

This procedure is of limited application when $W = (Z_k; p_k^z)$ and breaks down when $X^1 = (y_k^1; p_k^{y^1})$. The concept of marginal utility of wealth has to be redefined in these cases and we have suggested in section V a basis for such a redefinition. Our hypothesis does not provide a basis for measuring the marginal utility of uncertain incomes, but it would lead to:

1) a test of the proposition that there are diminishing returns to consumption in every period ($\lambda < 1$);

2) further specifications of indifference maps between various forms of income;

3) an ordering of uncertain prospects;

4) an extension of the theory of consumers' behavior to the case of uncertainty.

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