Can We Measure Marginal Utility by Observing

Choices Among Uncertain Prospects?

Jacques Dreze

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I. Definitions and notation

An uncertain prospect $X^1$ is a set of probabilities $p_i^1 \geq 0$ associated with mutually exclusive outcomes $x_i^1$, $i = 1, \ldots, n$, $\sum_i p_i^1 = 1$.

The possible outcomes $x_i^1$ are called the element of $X^1$; for convenience, we assume that they are expressed in money-terms, all real goods being reduced to monetary equivalents. This assumption is not essential, but simplifies the analysis.

The sum of money $x_i^1$ made available at the beginning of period $t$ is written $x_{t,1}^1$. The probability that $x_i^1$ will be made available in period $t$ is written $p_{t,1}^1$.

We represent $X^1$ by $(x_1^1, \ldots, x_n^1; p_1^1, \ldots, p_n^1)$ or $(x_i^1; p_i^1)$.

Given two independent uncertain prospects $X^1$ and $X^2$, the probability distribution of the sum of their elements is represented by $(x_1^1 + x_1^2, \ldots, x_n^1 + x_n^2)$; $p_1^1 p_2^1, \ldots, p_{1,1}^2 p_{1,2}^2, \ldots, p_{n,1}^1 p_{n,2}^2$ or $(x_1^1 + x_j^2; p_1^1 p_j^2)$.
The prospect \( x^1 = (x^1_{11}, x^1_{21}, \ldots, x^1_{s1}; p^1_{11}, p^1_{21}, \ldots, p^1_{s1}) \) is written
\( (y^1_k; p^1_k) \) where \( y^1_k \) stands for a particular stream of outcomes and \( p^1_k \) for
the probability of occurrence of that particular stream; \( i, j, \ldots, \ell = 1, \ldots, n; k = 1, \ldots, n^s \).

For \( p^1_k = 1 \), the present value of \( y^1_k = z^1_k \).

The uncertain prospect consisting of the probabilities \( r, 1 - r \) and of
the elements \( x^1, x^2 \) is represented by \([x^1_{i1}, x^1_{i2}; p^1_i, p^2_i(1 - i)]\).

The wealth, or net worth, \( W \), of an individual, is defined in terms of
the future returns to his assets and abilities. Superscripts are used to
distinguish various wealth prospects.

When the future returns are known with certainty, we write \( w_t \) for the
returns in period \( t \), \( \bar{w}_t \) for the present value of \( w_t \), and \( W = \sum_{t=1}^{s} \bar{w}_t \).

When the future returns are uncertain, \( v_t = (v_{t1}; p^W_{t1}) \) and
\( W = (v_{11} + v_{21} + \ldots + v_{s1}; p^W_{11}, p^W_{12}, p^W_{1\ell}, \ldots, p^W_{s1}, p^W_{s2}, p^W_{s\ell}) = (z_k; p^W_k) \) when \( z_k \) stands for a
particular income stream and \( p^W_k \) for the probability of occurrence of that
particular income stream. \( i, j, \ldots, \ell = 1, \ldots, n; k = 1, \ldots, n^s \)

For \( p^W_k = 1 \), the present value of \( z_k = \bar{z}_k \).

\( \mu \) and \( \phi \) stand for the relations of indifference and preference among
prospects.

Given a prospect \( x^1 \) and an arbitrary function \( U \), we write
\( \Sigma p^1_{i1} U(x^1_{i1}) = \mu^1_{i1}; \Sigma p^1_{i1} [U(x^1_{i1}) - \mu^1_{i1}]^2 = \mu^2_{i1}; \Sigma p^1_{i1} [U(x^1_{i1}) - \mu^1_{i1}]^3 = \mu^3_{i1} \).

Additional notation is introduced in Section V.
II. Bernoulli approach

1) There exists a utility function $U$

2) $U = U(W)$

3) $U = \log W$

4) $x^1 \preceq x^2$ if and only if $\sum p_i^1 \log (w + x_i^1) - \log w = \sum p_i^2 \log (w + x_i^2) - \log w$

$x^1 \preceq x^2$ if and only if $\sum p_i^1 \log (w + x_i^1) - \log w > \sum p_i^2 \log (w + x_i^2) - \log w$

Comments

1) The ordering of uncertain prospects is invariant under linear transformations of $U$, but only under such transformations.

2) If $x^1 = (x_1^1; p_1^1), x^2 = (x_1^2; p_1^2), W = (z_k; p_k^w)$, then:

$U(W) = \sum p_k^z \log \bar{z}_k$, and

$x^1 \preceq x^2$ if and only if $\sum_k \sum_i p_k^z p_i^1 \log (\bar{z}_k + x_i^1) - \sum p_k^z \log \bar{z}_k$

$= \sum_k \sum_i p_k^z p_i^2 \log (\bar{z}_k + x_i^2) - \sum p_k^z \log \bar{z}_k$

3) If $x^1 = (x_1^1; p_1^1), x^2 = (x_2^2; p_2^1), W = (z_k; p_k^w)$, then:

$x^1 \preceq x^2$ if and only if

$\sum_k \sum_j p_k^z p_j^1 \log (\bar{z}_k + \bar{y}_j^1) - \sum p_k^z \log \bar{z}_k = \sum_k \sum_j p_k^z p_j^2 \log (\bar{z}_k + \bar{y}_j^2) - \sum p_k^z \log \bar{z}_k$

4) Such a theory should be tested empirically. If it fails to yield accurate predictions, this may be due to several reasons, and in particular:

$U = U^*(W) + \log W$

$U = U(\mu_1^W, \mu_2^W, \mu_3^W)$
III. Merecksk approach

Basic postulates:

1) There exists a complete, transitive ordering of uncertain prospects.

2) That ordering is continuous in the probability space

3) \( X^1 \succ X^2 \) for some \( X^1, X^2 \) such that some \( 0 < p_i^1 < 1, 0 < p_j^2 < 1 \).

4) if \( X^1 \succ X^2 \), then, for any \( X^3 \)

\[ [x_i^1, x_j^3 ; p_i^1 r, p_j^3 (1-r)] \succ [x_i^2, x_j^3 ; p_i^2 r, p_j^3 (1-r)] \]

Theorem: There exists an \( f \), defined up to a linear transformation, such that:

\( X^1 \succ X^2 \) if and only if \( \Sigma p_i^1 f(x_i^1) = \Sigma p_i^2 f(x_i^2) \)

\( X^1 \succeq X^2 \) if and only if \( \Sigma p_i^1 f(x_i^1) > \Sigma p_i^2 f(x_i^2) \)

Comments:

1) This theory could be reformulated as follows:

Given postulates 1-3, the hypothesis that there exists an \( f \) such that

\( X^1 \succ X^2 \) if and only if \( \Sigma p_i^1 f(x_i^1) = \Sigma p_i^2 f(x_i^2) \), if true,

implies that \( f \) is defined up to a linear transformation, and that
postulate 4 holds. Plausibility arguments in favor of the hypothesis:

-- postulates 1-3 are merely consistency requirements
-- postulate 4, which is equivalent with the hypothesis, seems
  reasonable.

2) Given postulates 1-3 and \( W = \Sigma \bar{w}_t \) make the hypothesis that there

exists an \( f \) such that

\( X^1 \succ X^2 \) if and only if

\( \Sigma p_i^1 f(W + x_i^1) = \Sigma p_i^2 f(W + x_i^2) \).

If this hypothesis is true, the hypothesis that \( f \) is identical with
\( U^* \), up to a linear transformation, is not disproved. However, there
is no proof of that identity (the proof by Allais is incorrect).
3) If \( x^1 = (x^1_1; p^1_1), x^2 = (x^2_1; p^2_1), W = (z_k; p^z_k) \), then, given postulates 1-4, the theorem may become either:

A. There exists an \( f \), defined up to a linear transformation, such that

\[
x^1 \parallel x^2 \text{ if and only if } \sum_k \sum_i p^z_k p^1_i f(z_k + x^1_i) = \sum_k p^z_k p^2_i f(z_k + x^2_i)
\]

or:

B. There exists an \( f \), defined up to a linear transformation, such that

\[
x^1 \parallel x^2 \text{ if and only if } \sum_k p^1_i f(x^1_i) = \sum_k p^2_i f(x^2_i), \text{ given } W = (z_k; p^z_k)
\]

4) If \( x^1 = (y^1_k; p^1_k), x^2 = (y^2_k; p^2_k), W = (z_k; p^z_k) \), then, given postulates 1-4, the theorem may become either:

A. There exists an \( f \) defined up to a linear transformation, such that

\[
x^1 \parallel x^2 \text{ if and only if } \sum_k \sum_j p^z_k p^j_1 f(z_k + y^1_j) = \sum_k p^z_k p^j_2 f(z_k + y^2_j)
\]

or:

B. There exists an \( f \), defined up to a linear transformation, such that

\[
x^1 \parallel x^2 \text{ if and only if } \sum_k p^j_1 f(y^1_k) = \sum_k p^j_2 f(y^2_k), \text{ given } W = (z_k; p^z_k)
\]

IV. Allaiss approach

1) There exists a utility function, \( U \)

2) \( U = U(W), \frac{\partial U}{\partial W} > 0, \frac{\partial^2 U}{\partial W^2} < 0 \)

[U could be defined up to a linear transformation by means of introspection and psychological observations].
3) There exists an \( f \) such that
\[
x^1 I x^2 \text{ if and only if } f(\mu^1_1, \mu^1_2) = f(\mu^2_1, \mu^2_2)
\]

4) \[
\frac{\partial f}{\partial \mu_1} > 0, \quad \frac{\partial f}{\partial \mu_2} < 0
\]

5) when \[
\frac{\partial f}{\partial \mu_2} = 0, \quad x^1 I x^2 \text{ if and only if } \mu^1_1 = \mu^2_1 = \Sigma p^1_1 u(x^1_1) = \Sigma p^2_2 u(x^2_1)
\]

Comments:

1) Writing \( W^1 \) for \((W + x^1)\) and \( W^2 \) for \((W + x^2)\), this should be restated:

There exists an \( f \) such that
\[
x^1 I x^2 \text{ if and only if } f(\mu^1_1, \mu^1_2) = f(\mu^2_1, \mu^2_2)
\]

\[
\frac{\partial f}{\partial \mu^1_1} > 0, \quad \frac{\partial f}{\partial \mu^2_2} < 0
\]

When \[
\frac{\partial f}{\partial \mu_2} = 0, \quad x^1 I x^2 \text{ if and only if }
\]
\[
\mu^1_1 = \mu^2_1 = \Sigma p^1_1 u(W + x^1_1) = \Sigma p^2_2 u(W + x^2_1)
\]

2) When \( W = (Z_k; p^k_k) \),
\[
\mu^1_1 = \Sigma \Sigma p^k_k p^1_1 u(Z_k + x^1_1) + \mu^W_1 + \mu^1_1
\]
\[
\mu^1_2 = \Sigma \Sigma p^k_k p^1_1 [u(Z_k + x^1_1) - \mu^1_1]^2 + \mu^W_2 + \mu^1_2
\]
3) Why not introduce $\mu_2$ as well as $\mu_1$ and $\mu_2$ into the analysis?

4) if $\frac{\partial e}{\partial \mu_2} < 0$, does it follow that: $\frac{\partial e}{\partial \mu_1} < 0$? $\frac{\partial e}{\partial \mu_1} < 0$?

V. The problem of present values of future incomes.

Write $c_t$ for the consumption and $s_t$ for the savings of an individual in period $t$, $t = 1, \ldots, s$.

When $W = \sum \bar{w}_t$, $c_t = c_t(W)$

When $W = (z_i; P_i^s)$, $c_t = [c_{ti}(W); p_{ti}]$ $t = 1, \ldots, s$

$i = 1, \ldots, n$

Hypothesis:

a) There exists a $V$, defined up to a monotonic transformation, and

a $v$, defined up to a linear transformation, such that:

$V^1 I W^2$ if and only if

$v^1 \left( \sum_{i=1}^{s} p_{i1}^2 v(c_{i1}^1(W^1)) \ldots, \sum_{i=1}^{s} p_{i1}^2 v(c_{i1}^1(W^1)) \right)$

$= v \left( \sum_{i=1}^{s} p_{i1}^2 v(c_{i1}^2(W^2)) \ldots, \sum_{i=1}^{s} p_{i1}^2 v(c_{i1}^2(W^2)) \right)$

b) $V$ is homogeneous of some positive degree, different from 1, in $c_{ti}$;

$t = 1, \ldots, s$

$i = 1, \ldots, n$

The second part of the hypothesis is equivalent to assumption II of Kodigliani and Brumberg. It implies that $V$ be homogeneous in $v$, and $v$ in $c_{ti}$. Without loss of generality, we may write $v(c_{ti}) = \tau c_{ti}^\lambda$, $\lambda > 0$, $\lambda \neq 1$, $\tau > 0$. 
**Simplified model**

\( s = 2; \) no rate of interest, no carry-over of savings beyond the second period, no uncertainty attached to the needs or to the value of the savings. \( W = (Y_1 + Y_{21}; p_1) \)

\[ V = \alpha c_1^\lambda + \beta \sum p_1 c_{21}^\lambda + \gamma \sum p_1 [\beta c_{21}^\lambda - \beta (r_{21} c_1)^\lambda] \]

\[ = (\alpha - \gamma \beta r_{21}^\lambda) c_1^\lambda + \beta (1 + \gamma) \sum p_1 c_{21}^\lambda \]

\[ = (\alpha - \gamma \beta r_{21}^\lambda) c_1^\lambda + \beta (1 + \gamma) \sum p_1 (Y_1 + Y_{21} - c_1)^\lambda \]

\( \gamma \geq 0, \)

\( r_{21} \) is the ratio \( \frac{c_2}{c_1} \) which would be achieved under conditions of certainty. That ratio is fixed as a consequence of our assumption of homogeneity. \( V \) is (tentatively!) written in an additive form:

1) to make the degree of its homogeneity in \( c_1, c_{21} \) independent of the arbitrary choice of a time-unit without any further assumption regarding the change in \( \lambda \) induced by a change in the length of the time-unit.

2) on the ground that, under conditions of certainty, monotonic transformations on \( V \) can account for the interdependence between \( c_1 \) and \( c_2 \). The term \( \gamma \sum p_1 [\beta c_{21}^\lambda - \beta (r_{21} c_1)^\lambda] \) corresponds to the idea that an expectation of improvement in the consumption level is a form of present consumption.

Since under conditions of certainty \( \frac{\partial V}{\partial c_1} = \frac{\partial V}{\partial c_2} = \lambda \alpha c_1^{\lambda-1} = \lambda \beta c_2^{\lambda-1}, \frac{\lambda \alpha}{\lambda \beta} = \frac{(c_2^\lambda)}{c_1^{\lambda-1}} \) and \( \frac{\alpha}{\beta} = r_{21}^{\lambda-1} \); set \( \alpha = 1 \) (choice of unit).
Then $\beta = \frac{1}{r_{21} - \lambda}$ and

(I) \[ V = (1 - \gamma r_{21}) c_1^\lambda + \beta (1 + \gamma) \Sigma p_1 (y_1 + y_{21} - c_1)^\lambda \]

Maximizing (I) with respect to $c_1$, we get:

\[ \frac{\partial V}{\partial c_1} = \lambda (1 - \gamma r_{21}) c_1^{\lambda-1} - \lambda \beta (1 + \gamma) \Sigma p_1 (y_1 + y_{21} - c_1)^{\lambda-1} = 0 \]

\[ c_1^{\lambda-1} = \frac{\beta (1 + \gamma)}{1 - \gamma r_{21}} \Sigma p_1 (y_1 + y_{21} - c_1)^{\lambda-1} \]

or \[ (y_1 - c_1)^{\lambda-1} = \frac{\beta (1 + \gamma)}{1 - \gamma r_{21}} \Sigma p_1 (y_{21} + s_1)^{\lambda-1} \]

For $\beta = 1$, $\lambda < 1$, $\Sigma p_1 y_{21} = y_1$, $\gamma = 0$, this equation will be satisfied only by positive values of $s_1^*$ and $\frac{\partial s_1^*}{\partial \beta} > 0$, $\frac{\partial s_1^*}{\partial \gamma} > 0$.

Extension of the model

A. 3-periods case: $a = 3, \gamma = 0; W = (y_1 + y_{21} + y_{31} + p_{21} + p_{31})$

\[ V = \alpha c_1^\lambda + \beta \Sigma p_{21} c_{21}^\lambda + \delta \Sigma p_{31} c_{31}^\lambda \]

If $c_1$ and $y_{21}$ are given, we may write:

\[ V(c_{21}, c_3 | c_1, y_{21}) = \beta c_{21}^\lambda + \delta \Sigma p_{31} [y_{3j} + y_{21} + y_1 - c_1 - c_{21}]^\lambda \]

For a given $c_1$

\[ V(c_2, c_3 | c_1) = \Sigma p_{21} V(c_{21}, c_3 | c_1, y_{21}) = \beta \Sigma p_{21} c_{21}^\lambda + \delta \Sigma p_{21} p_{3j} [y_{3j} + y_{21} + y_1 - c_1 - c_{21}]^\lambda, \]

and

\[ V(c_1, c_2, c_3) = \alpha c_1^\lambda + \beta \Sigma p_{21} c_{21}^\lambda + \delta \Sigma p_{21} p_{3j} [y_{3j} + y_{21} + y_1 - c_1 - c_{21}]^\lambda \]
The complete formulation \((\gamma \neq 0)\) would be:

\[ v = \alpha c_1^\lambda + \beta \sum p_{21} c_{21}^\lambda + \gamma \sum p_{21} (\beta c_{21}^\lambda - \beta r_{21}^\lambda c_1^\lambda) + \gamma \sum p_{31} c_{31}^\lambda + \varepsilon \sum p_{31} (5c_{31}^\lambda - 8 r_{31}^\lambda c_1^\lambda) + \xi \sum p_{21} p_{3j} (5c_{3j}^\lambda - 8 r_{3j}^\lambda c_{21}^\lambda) \]

B. Uncertainty about future needs: \(\beta\) becomes \((\beta_1; p_1^\beta)\)

Uncertainty about the value of savings: \(c_{21}\) becomes \(\sum p_{21} p_j^s (y_{21} + a_j s_1)\)

In this case, even with \(\beta = 1, \lambda < 1, \sum p_{21} y_{21} = y, \gamma = 0, \sum p_j^s a_j s_1 = s_1,\) \(s_1^s\) may well be negative.

Problem of allocation of savings between purchases of durable goods and other forms of investment: write \(\vartheta_1\) for the savings invested in the form of durable, \(S_1\) for the savings invested in other forms, and \(a_1 Y_1\) for \(X_{21}\).

It seems reasonable to assume then that \(c_{21} = a_1 (Y_1 + S_1) + S_1\) -- or eventually

\[ c_{21} = a_1 (Y_1 + b_1 \vartheta_1) + S_1 \]

A special treatment of savings in the form of insurance policy should also be considered.

VI. Conclusions

Only the Marschak approach provides a basis for systematic measurement of the marginal utility of wealth.

This procedure is of limited application when \(W = (Z_k; p_k^s)\) and breaks down when \(X^1 = (y_k^1; p_k^1)\). The concept of marginal utility of wealth has to be redefined in these cases and we have suggested in section V a basis for such a redefinition. Our hypothesis does not provide a basis for measuring the marginal utility of uncertain incomes, but it would lead to:
1) a test of the proposition that there are diminishing returns to consumption in every period \( \lambda < 1 \);

2) further specifications of indifference maps between various forms of income;

3) an ordering of uncertain prospects;

4) an extension of the theory of consumers' behavior to the case of uncertainty.

References


Modigliani, F. and Brumberg, R., Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data, forthcoming.