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Distribution of Completion Times for Random Communication in a Task-Oriented Group

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1. Introduction

One of the outcomes of Kurt Lewin's suggestions toward the group dynamics approach to psychology has been experiments on cooperative problem solving by small groups of people. The type of experiments with which we are particularly concerned here have been reported by Bavelas (1950) and Christie, Luce and Macy (1952, this will be referred to as CLM).

The basic question of how people behave when they must work together to solve a problem or complete a task is, of course, tremendously complicated and in these experiments attention is directed mainly to one aspect: the effect of various restrictions on the possibilities for communication between individuals. The attempt is made to minimize the complexities due to such factors as individual differences in intelligence and personality, differences in motivation, etc. by making the task of trivial simplicity so that no ingenuity is required and there can be little emotional involvement. A typical task is as follows. Each individual is given a set of colored marbles, only one color being common

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to all the group. The members of the group must exchange messages about their own colors and what they have learned about the colors of the others, until finally everybody knows the common color. To eliminate the effects of differences in speed of perception and response, messages are sent only after everyone has indicated readiness to transmit; the transmissions then take place simultaneously, each individual sending to just one of his possible recipients. Each member of the group knows initially to whom he can send messages, but does not know to whom the others can send.

In the following we consider some mathematical questions arising from these experiments. One of the main aims is to determine what type of communication net (i.e. restrictions on the paths for messages) is optimum for reducing the completion time, T, i.e., the time required to complete the task. This question is not answered here, but it is hoped that the present considerations, will provide a start toward the answer. Of course, even the greatly simplified simulation of the group problem-solving situation which the above experiments represent is still too complex for a complete mathematical treatment; important aspects must be omitted and further simplifying assumptions are needed. Thus the questions of group learning -- to which much attention is devoted in CLM -- and of morale and satisfaction are not included in this treatment.

The major assumption here is that messages are sent at random; that is, at each sending time each individual picks a recipient from among those to whom he is permitted to send messages with equal probability* of picking any one of them. This is, of course, not a correct statement of how people will behave in small task-oriented groups, because it means that the same information might be repeated to a recipient, thereby failing to communicate new information, instead of deliberately choosing a recipient to whom the sender's information

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* The assumption of equal probability is not essential as noted in Sec. 6.

may be new. In spite of the obvious objections to this assumption of randomness it seems worthwhile to consider this problem for the following reasons.

In the first place, although the assumption of random choice of recipient can not be exactly correct, it does seem to approximate the actual behavior of the group at first, if the communication net is fairly complex. This is indicated by the results reported in CLM. The distribution of completion times under the assumption of random sending (abbreviated to RD, random distribution, in the following) was computed approximately by a Monte Carlo method in connection with these experiments. It can be seen (Chap. III of CLM) that for the more complicated communication structures the observed distribution of completion times for the first six trials was not too far from the RD. Of course, there was learning with increasing experience, so that later trials showed improvement over randomness. With simple communication structures the observed times departed considerably from the RD right from the start, but even in these cases the RD gave a rough indication of what was to be expected.

The last remark points to a second use for the RD. This is as a basis for comparison, an indication of what to expect in any case. It is obviously of value to have such a basis even though divergence from it is expected. The divergence itself has significance as an indication of rational rather than random behavior.

Finally the assumption of randomness can be justified on the principle that difficult problems can often be solved by treating simplified versions of the problem first and then adding the additional complicating factors needed to bring the situation closer to reality. The alternatives to the random assumption would require considerations about what rational behavior would be for each individual at each stage, and also to what extent the actual individuals involved can be expected to be rational. These are obviously questions which are not easy to answer. The only relaxation of randomness which might not present too much difficulty is the minimum assumption that a sender will never repeat a message when he is certain that carries no new information to the recipient.

2. Minimum completion time.

We consider first a question not involving the assumption of random sending. For a group of given size, n, and any possible arrangement of the communication net, what is the minimum pcompletion time assuming now that the individuals are told where to send their messages so as to achieve this minimum. This minimum, time τ, is the best that can be done under any conditions given n and the number of messages, r, which each individual can send at each time. For the usual case of one message at each time, the value of τ has often been stated but the writer does not know of any published proof. The minimum time, τ, is in this case the smallest integer containing $\log_2 n$. If we allow more than one, say r, messages to be sent by each individual at each time, then τ is the smallest integer containing $\log_{r+1} n$, which is the same as stating that τ is the integer satisfying

$$\log_{r+1} n \leq \tau < 1 + \log_{r+1} n \tag{2.1}$$

$$\text{or} \quad n \leq (r+1)^{\tau} < (r+1)\, n$$

A proof of this result is given in the appendix.

3. Notation

From now on we consider the problem of Sec. 1, the determination of the RD. To describe the structure of the communication net and the flow of information we use matrix notation (c.f. Shimbel, 1951, and CLM).

Let the individuals be numbered $1, 2, \ldots n$. If individual i can send directly to j we say there is a link from i to j; this is an outgoing link for i and an incoming link for j. The structure of the communication net can be given by a matrix S with elements s_{ij} where

$$\begin{aligned} & s_{ii} = 1 \\ \text{and for } i \neq j, \quad & s_{ij} = 1 \text{ if there is a link from } i \text{ to } j, \\ & s_{ij} = 0 \text{ if there is no link from } i \text{ to } j. \end{aligned} \tag{3.1}$$

The units in the principal diagonal describe the fact that each individual remembers all the information he has received.

We assume that S is connected, which means that for every ordered pair i,j there is a path from i to j; that is, a set of non-zero elements $s_{ii_1}, s_{i_1 i_2}\, s_{i_2 i_3} \ldots s_{i_k j}$. This is equivalent to stating that some power of S has no zero elements. The interpretation of connectedness is that completion of the task is possible, since there is a chain of individuals who can pass a message from any i to any j.

At each sending time, each individual chooses one recipient for his message, that is, one of his outgoing links. These n choices give a sending pattern which can be specified by one of the set of sending matrices, $S^{(v)}$, $v=1,2,\ldots n_s$. The elements of $S^{(v)}$ are $s_{ij}^{(v)}$ where

$$\begin{aligned} & s_{ii}^{(v)} = 1, \\ \text{and for each } i, \quad & s_{ij}^{(v)} = 1 \text{ for one of the } j\text{'s for which } s_{ij} = 1 \quad (j \neq i), \\ \text{and} \quad & s_{ij}^{(v)} = 0 \text{ for all other } j \neq i. \end{aligned} \tag{3.2}$$

The assumption of random choice of outgoing links with equal probability for each means that, the probability that $s_{ij}^{(v)} = 1$ (for $i \neq j$) is $s_{ij}/(\sum_j s_{ij} - 1)$, since the denominator here is the number of i's outgoing links. The number of possible sending matrices $S^{(v)}$ is $n_s = \prod_{i=1}^{n} (\sum_j s_{ij} - 1)$. (3.3)

Since the choices of the n individuals are independent, the probability of any one of these is $^1/n_s$.

The distribution of information among the individuals of the group at any time is specified by one of the information state matrices $C^{(\alpha)}$ with elements $c_{ij}^{(\alpha)}$ where

$$c_{ij}^{(\alpha)} = 1, \text{ if } i\text{'s initial information is known to } j \qquad (3.4)$$
$$c_{ij}^{(\alpha)} = 0, \text{ otherwise.}$$

At the start, the information state is given by $C^{(0)} = I$, the identity matrix; and at the completion time, the information state is given by a matrix whose elements are all unity, we call this $C^{(N)}$. It is possible to write down $2^{n(n-1)}$ information state matrices, corresponding the number of ways of placing o's and 1's in the $n(n-1)$ off-diagonal positions. However, not all of these can actually be attained because of the restrictions imposed by the $S^{(v)}$.

With this notation the information state which results if we have the state $C^{(\alpha)}$ and send messages according to the matrix $S^{(v)}$ is the matrix product $C^{(\alpha)}S^{(v)}$. In forming this product, Boolean arithmetic is used*, which means that $1 + 1 = 1$, the other rules of addition and multiplication being as usual. It is easy to see that the i,j-th element of $C^{(\alpha)}S^{(v)}$ which is $\sum_k c_{ik}^{(\alpha)} s_{kj}^{(v)}$ will be 1 if and only if there is at least one value of k for which $c_{ik}^{(\alpha)} = 1$ and $s_{kj}^{(v)} = 1$, but this means that i's initial information is known to some individual, k, and k sends to j.

We also need the quantity of information in the information state matrix $C^{(\alpha)}$. This is given by

$$I_{(\alpha)} = \sum_{i,j=1}^{n} c_{ij}^{(\alpha)},$$

and simply counts the total number of pieces of information known to all the

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* Boolean arithmetic is used only for these matrix products in this paper.

members of the group.

4. Markov chain formulation.

To treat the problem of calculating the RD we consider the $C^{(\alpha)}$ as the states of a finite Markov chain with stationary transition probabilities (Feller, 1950). We denote the transition probabilities by $a_{\alpha\beta}$, this being the probability that when the information state is $C^{(\alpha)}$, it becomes $C^{(\beta)}$ after one sending time. These transition probabilities are easily calculated as follows: form the products $C^{(\alpha)}S^{(1)}$, $C^{(\alpha)}S^{(2)}, \ldots, C^{(\alpha)}S^{(n_s)}$, if n_β of these products are $C^{(\beta)}$, then $a_{\alpha\beta} = \frac{n_\beta}{n_s}$.

The matrix of transition probabilities $(a_{\alpha\beta})$ can be made to assume a particularly simple form, in fact, it can be written as a triangular matrix, i.e. all elements below the principal diagonal are zeros. This can be seen by considering I_α. From the fact that $s_{jj}^{(v)} = 1$ for all j and v it follows that if $C^{(\beta)} = C^{(\alpha)}S^{(v)}$, so that $c_{ij}^{(\beta)} = \sum_k c_{ik}^{(\alpha)} s_{kj}^{(v)}$ we have $c_{ij}^{(\beta)} = 1$ whenever $c_{ij}^{(\alpha)} = 1$, so that $I_{(\beta)} \geq I_{(\alpha)}$. Also a transition with $I_{(\beta)} = I_{(\alpha)}$ can occur only if $C^{(\beta)} = C^{(\alpha)}$. It is, of course, possible to have $I_\beta = I_\alpha$ with $C^{(\beta)} \neq C^{(\alpha)}$ but transitions between two such states can never occur. The only possible transitions are to the same state or one with a larger value of I_α, so that by assigning the indices α in order of increasing I_α (with any assignment for the $C^{(\alpha)}$ having equal I_α) the transition probability matrix becomes triangular.

It is now easy to characterize the states (Feller, 1953). The fact that structure is connected means that for every α except $\alpha = N$, there is a β with $\beta > \alpha$ and $a_{\alpha\beta} > 0$; for $\alpha = N$, $a_{NN} = 1$. Thus every state is transient, except the final state (task completed) which is the single absorbing state.

Let $a_{\alpha\beta}^{(t)}$ be the probability of a transition from $C^{(\alpha)}$ to $C^{(\beta)}$ in t

steps, so the $a_{\alpha\beta}^{(t)}$ are the elements of the t-th power of the matrix, $A = (a_{\alpha\beta})$. Then $a_{oN}^{(t)}$ is the probability of completion of the task at time t or earlier. The cumulative RD is then given by $a_{oN}^{(1)}, a_{oN}^{(2)}, \ldots,$ and these values can be obtained in succession by multiplying the row vectors $(a_{00}^{(t)}, a_{01}^{(t)}, \ldots, a_{0N}^{(t)})$ by A, i.e. $\left(a_{00}^{(t+1)}, a_{01}^{(t+1)}, \ldots, a_{0N}^{(t+1)}\right) = (a_{00}^{(t)}, a_{01}^{(t)}, \ldots, a_{0N}^{(t)})\, A.$ (4.1)

However, we can do more than this. Explicit expressions for the generating function of the RD and for the expected value of the completion time can be written. Let $p_\alpha(t)$ $t=1,2,\ldots,$ and $\alpha = 0,1,\ldots,N-1$ be the probability that starting from $C^{(\alpha)}$ the final state $C^{(N)}$ is reached in t steps and not fewer. Then

$$p_\alpha(1) = a_{\alpha N}, \quad \alpha = 0, 1, \ldots, N-1 \tag{4.2}$$

$$p_\alpha(t+1) = \sum_{\beta=\alpha}^{N-1} a_{\alpha\beta}\, p_\beta(t), \quad \text{for } t > 0. \tag{4.3}$$

Introducing the generating functions for the $p_\alpha(t)$,

$$G_\alpha(u) = \sum_{t=1}^{\infty} p_\alpha(t)\, u^t,$$

we obtain using (4.2) and (4.3)

$$\begin{aligned}\frac{1}{u} G_\alpha(u) &= a_{\alpha N} + \sum_{t=1}^{\infty} p_\alpha(t+1)\, u^t \\ &= a_{\alpha N} + \sum_{t=1}^{\infty} \sum_{\beta=\alpha}^{N-1} a_{\alpha\beta}\, p_\beta(t)\, u^t \\ &= a_{\alpha N} + \sum_{\beta} a_{\alpha\beta} \sum_{t} p_\beta(t)\, u^t\end{aligned}$$

so that $$\frac{1}{u} G_\alpha(u) - \sum_{\beta=\alpha}^{N-1} a_{\alpha\beta}\, G_\beta(u) = a_{\alpha N} \tag{4.4}$$

The set of equations in (4.4) can be solved in succession starting with the last

$$\left(\frac{1}{u} - a_{N-1,N-1}\right) G_{N-1}(u) = a_{N-1,N}$$

$$\left(\frac{1}{u} - a_{N-2,N-1}\right) G_{N-2} - a_{N-2,N-1}\, G_{N-1}(u) = a_{N-2,N}$$

$$\text{--} \qquad (4.5)$$

$$\left(\frac{1}{u} - a_{00}\right) G_0(u) - a_{01}\, G_1(u) - \ldots\ldots - a_{0,N-1}\, G_{N-1}(u) = a_{0N}.$$

We are mainly interested in $G_0(u)$, whose coefficients give the RD. From (4.4) or (4.5), we see that $G_0(u)$ can be written in either of the two explicit forms.

$$G_0(u) = \frac{a_{0N}}{\frac{1}{u} - a_{00}} + \sum_{\alpha=0}^{N-1} \frac{a_{0\alpha}}{\frac{1}{u} - a_{00}} \frac{a_{\alpha N}}{\frac{1}{u} - a_{\alpha\alpha}} + \sum_{0<\alpha<\beta}^{N-1} \frac{a_{0\alpha}}{\frac{1}{u} - a_{00}} \frac{a_{\alpha\beta}}{\frac{1}{u} - a_{\alpha\alpha}} \cdot$$

$$\frac{a_{\beta N}}{\frac{1}{u} - a_{\beta\beta}} + \ldots + \prod_{\alpha=0}^{N-1} \frac{a_{\alpha,\alpha+1}}{\frac{1}{u} - a_{\alpha\alpha}} \qquad (4.6)$$

$$G_0(u) = \frac{1}{\prod_{\alpha=0}^{N-1}\left(\frac{1}{u} - a_{\alpha\alpha}\right)} \begin{vmatrix} a_{01} & a_{02} & a_{03} & a_{04} & - & a_{0,N-1} & a_{0N} \\ a_{11}-\frac{1}{u} & a_{12} & a_{13} & a_{14} & - & a_{1,N-1} & a_{1N} \\ 0 & a_{22}-\frac{1}{u} & a_{23} & a_{24} & - & a_{2,N-1} & a_{2N} \\ 0 & 0 & a_{33}-\frac{1}{u} & a_{34} & - & a_{3,N-1} & a_{3N} \\ - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & a_{N-1,N-1}-\frac{1}{u} & a_{N-1,N} \end{vmatrix} \qquad (4.7)$$

If E_α is the expected value of the completion time starting from $C^{(\alpha)}$, so that $E_\alpha = G'_\alpha(1)$, then from (4.4) using $G_\alpha(1) = 1$ and $\sum_{\beta=\alpha}^{N} a_{\alpha\beta} = 1$, we obtain the equations for the E_α

$$E_\alpha - \sum_{\beta=\alpha}^{N-1} a_{\alpha\beta} E_\beta = 1, \quad \alpha = 0,1,\ldots, N-1. \tag{4.8}$$

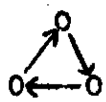
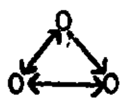
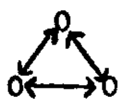
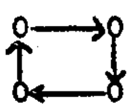
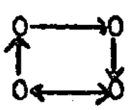
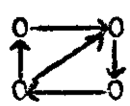
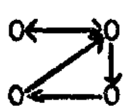
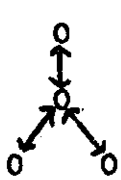
Then $E_0 = E(T)$, the expected completion time starting from the initial state $C^{(0)}$, is given by the expressions in (4.6) and (4.7) when u and all the $a_{\alpha N}$ are replaced by 1.

Equations for the second and higher moments can be obtained by taking higher derivates of $G_\alpha(u)$ using (4.4).

5. Calculations

In the following table are given the generating functions of the RD and the average completion time for all possible connected nets for $n = 3$ and a few for $n = 4$.

The values of $G_0(u)$ and $E(T)$ in the table were calculated using the formulas of Sec. 4, but in certain cases the work of calculation was shortened because it was possible to represent the information state more simply than by using the matrices $C^{(\alpha)}$. This simplification is discussed below.

Structure	$G_0(u)$	$E(T)$
n=3		
	u^2	2
	$\frac{u^2}{2(1-\frac{u}{2})}$	3
	$\frac{u^2}{16}\left(-16 + \frac{6}{1-\frac{u}{2}} + \frac{15}{1-\frac{u}{4}}\right)$	$3\frac{1}{6}$
	$\frac{u^2}{16}\left(-20 + \frac{27}{1-\frac{u}{4}}\right)$	$2\frac{3}{4}$
	$\frac{u^3}{2(1-\frac{u}{2})}$	4
n=4		
	u^3	3
	$\frac{u^2}{4}\left(-2 + \frac{3}{1-\frac{u}{2}}\right)$	$4\frac{1}{2}$
	$\frac{u^3}{2(1-\frac{u}{2})}$	4
	$\frac{u^3}{4}\left(-3 - u + \frac{4}{1-\frac{u}{2}}\right)$	$4\frac{3}{4}$
	$\frac{u^3}{16}\left(\frac{8}{(1-\frac{u}{2})^2} + \frac{8}{1-\frac{u}{4}} - \frac{15}{(1-\frac{u}{4})^2}\right)$	$6\frac{1}{9}$
	$\frac{2}{9}u^4\left(\frac{2}{1-\frac{2}{3}u} - \frac{1}{1-\frac{u}{3}}\right)$	$6\frac{1}{2}$

We have also calculated, using (4.1), the first six values of $p_0(t)$, for the linear chain with $n = 5$, i.e. the structure in Fig. 2 below. These are compared with the values reported in CLM which were computed by a Monte Carlo technique on the Whirlwind I computer using 3000 trials.

t	5	6	7	8	9	10
$p_0(t)$ (Eq. 4.1)	.070	.152	.189	.187	.127	.100
$p_0(t)$ (CLM)	.078	.149	.197	.180	.118	.103

The calculations by these methods get very lengthy as n increases but the procedure is straightforward and could easily be handled by a high-speed computing machine, at best for moderate n.

For some special structures the calculation can be simplified because, for the present purpose of finding the completion time, the information state can be specified by less than the complete set of $C^{(\alpha)}$ as defined above. For example, in the star (Fig. 1) it can be seen that we need only specify the number α of individuals who have all the information. The distribution of completion times for this net is easily computed directly, but it can also be derived as a simple case of the Markov chain formulation. The result is

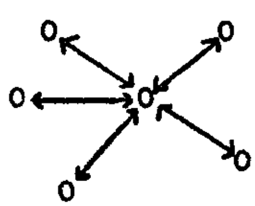


Fig. 1

$$G_0(u) = u^n (n-2)! \prod_{\alpha=1}^{n-2} \frac{1}{n-1-\alpha u} \tag{5.1}$$

$$E(T) = 1 + (n-1) \sum_{\alpha=1}^{n-1} \frac{1}{\alpha} \tag{5.2}$$

Another example is the linear chain. The information state here can

$$\underset{1}{0} \longleftrightarrow \underset{2}{0} \longleftrightarrow \underset{3}{0} \longleftrightarrow 0 \cdots\cdot 0 \longleftrightarrow \underset{n}{0}$$

Fig. 2

be specified by two numbers (α, β) where α is the number of individuals who have received 1's information and β is the number who have n's information. This specification is sufficient because we are only interested in completion time, so that if 2 sends to 3 before receiving 1's information he must send again after receiving from 1. If, as mentioned below, we are interested in the time to reach an information state less than the completion state, this simplified specification may not be sufficient.

A further simplification of the linear chain is possible because reversing the numbering of individuals gives an equivalent state; that is, (α,β) can be identified with (β,α). In general, symmetries in the structure can be used to reduce the number of information states which need to be considered. Thus in the structure where each individual has a link to each of the others, two information state matrices may be treated as identical if one results from the other by applying the same permuaation to its rows and columns.

6. Generalizations and further problems.

The assumption that each individual chooses from his outgoing links with equal probability can be replaced by less restrictive assumptions without any essential change in the method. Thus instead of equal probability we could assign any fixed probabilities to each individual's choices of links, or we could assign arbitrary probabilities to the sending matrices $S^{(v)}$, or the probability of any $s_{ij}^{(v)}$ or $S^{(v)}$ could even depend on the preceding information state $C^{(\alpha)}$. In any of these cases the transition probabilities $a_{\alpha\beta}$ are determined by $C^{(\alpha)}$, and this is all that is required in order to have a simple Markov chain.

However, the introduction of a rule like, i does not repeat a message to j if it contains nothing new, does make an essential change. In this case $a_{\alpha\beta}$ is not determined by a knowledge of $C^{(\alpha)}$ but depends also on how $C^{(\alpha)}$ was reached. **This process is not a simple Markov chain.**

In some problems of communication in networks, the question is not the time to reach the complete state but a specified information state, say C^*. This can be treated by the method of Sec. 4, if we identify with C^* all the $C^{(\alpha)}$ which include C^*, i.e. for which $c_{ij}^{(\alpha)} = 1$ if $c_{ij}^* = 1$. To insure the proper ordering, we must count the information content of C^* as being n^2. In this case, some of the simplifications of Sec. 5 may not be available. The case where we do not start with $C^{(0)} = I$ but with some other information state, $C^{(\alpha)}$ is also included above; in fact, we have the formulas for $G_\alpha(u)$ and $E_\alpha(T)$.

It was hoped that this method would give information about the structures which minimize $E(T)$, but this problem will require further study. The values for small n which we have computed are probably not a very good indication of what happens as n increases because of the rapidity with which the number of possible structures increases. However, an extensive list of examples worked on a high-speed computing machine might indicate where to look for the best structures.

Appendix

Minimum Completion Time

The problem here is how to arrange the net and the sending schedule so as to achieve the shortest possible completion time, τ, subject only to the limitation that each individual sends not more than a fixed number, r, of messages at every sending time. We show that

$$\tau = \text{smallest integer containing } \log_{r+1} n, \tag{A1}$$

which is the same as stating that τ is the integer satisfying

$$\log_{r+1} n \leq \tau < 1 + \log_{r+1} n \tag{A2}$$

$$\text{or} \quad n \leq (r+1)^{\tau} < (r+1)\, n$$

It is easy to see that τ can not be smaller than the value given by (A1) as follows. Let $I(t)$ be the total information possessed by the group after sending time t. We have $I(0) = n$, and, if T is the completion time, $I(T) = n^2$. Now after any sending time $I(t)$ can be at most $r + 1$ times its value at $t - 1$, or

$$I(t) \leq (r + 1)\, I\,(t - 1) \tag{A3}$$

This is because the maximum amount of new information which each individual can give to the other members of the group by his r messages at time t is r times the information he had at time $t - 1$. Since this is true for every member of the group, $I(t) \leq I(t - 1) + r\, I\,(t - 1)$, which is (A3). The equality in (A3) holds only if every message send consists entirely of information which is new to the recipient, if there is any repetition of already known information then the inequality holds. From (A3)

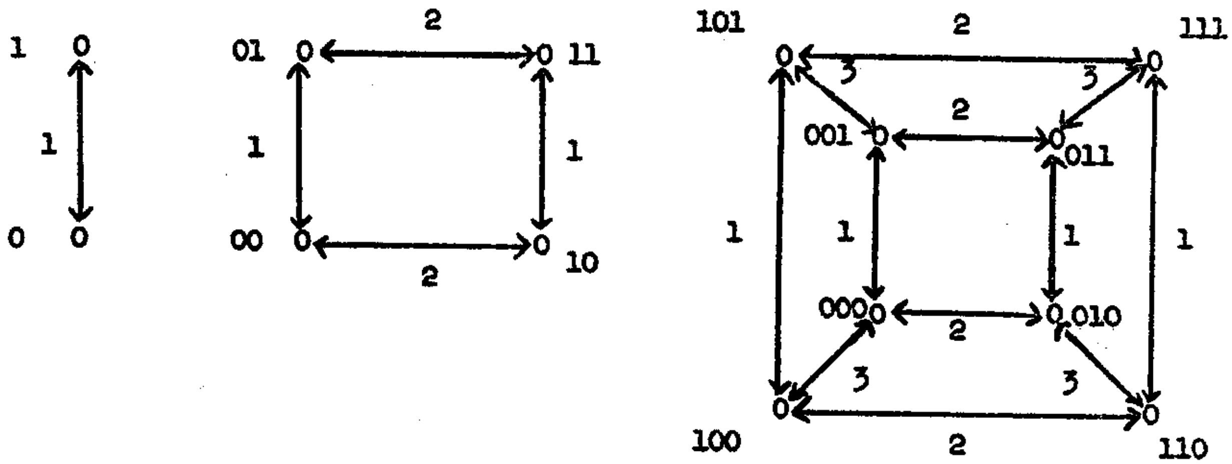


Fig. 3. Communication nets giving minimum completion time for $n = 2, 4, 8$, and $r = 1$.

$$I(t) \leq (r + 1)^t \, I(0) = n(r + 1)^t$$

and using $I(T) = n^2$,

$$n \leq (r + 1)^T$$

or $$T \geq \log_{r+1} n;$$

and since $\tau = \min T$,

$$\tau = \log_{r+1} n. \tag{A4}$$

Equation (A1) says that the equality in (A4) is attained if n is a power of $r + 1$. To see this consider first the case $r = 1$ and let $n = 2^m$. In Figure 3 we have graphed the communication nets which give the minimum completion time for $n = 2$, 4 and 8. The numbers on the lines connecting the points show the paths over which messages are sent at each time. It is clear that the indicated pairing-off arrangement can be continued to any power of 2. Formally what we do is give each individual a number from 0 to $n - 1$ and write the number in binary notation; i.e. i is written as

$$e_{im} \, e_{i,m-1} \cdots e_{i2} \, e_{i1} \tag{A5}$$

where each e is either 0 or 1 and (A5) means

$$i = e_{i1} + 2e_{i2} + 2^2 e_{i3} + \ldots + 2^{m-1} e_{im} \tag{A6}$$

The $n = 2^m$ individual require all the binary number that can be formed using m binary digits. The instructions for sending messages, so as to attain the minimum solution time are then that at sending time t, each individual sends to that individual whose binary number is the same as his <u>except</u> in the t-th digit; that is, i sends that j for which

$$e_{ik} = e_{jk}, \quad \text{for } k = 1,2,\ldots, t-1, t+1, \ldots m \tag{A7}$$
$$e_{it} \neq e_{jt}.$$

It can now be seen that in at least m steps, the initial information of any i will be transmitted to any j. Suppose i's binary number differs from j's in the m_1-th, m_2-th, ..., m_k-th digits

$$e_{im_1} \neq e_{jm_1}, e_{im_2} \neq e_{jm_2}, \ldots, e_{im_k} \neq e_{jm_k},$$

and i and j have the same digits in the other positions. Then at time m_1, i sends to i_1 having the same binary number as i except that $e_{i_1m_1} = e_{jm_1}$; at time m_2, i_1 sends to i_2 with the same binary number as i_1 except $e_{i_2m_2} = e_{jm_2}$; continuing in this way i's information reaches j at time m_k. This proves the equality in (A2) for $r = 1$ and n a power of 2.

Now if $n = 2^m-1$, we use binary numbers as above except one number is omitted, say 0. The sending instructions are the same except when the recipient is 0 according to (A7) the instructions are modified so that now the recipient is the individual to whom 0 would have sent at the following time. Thus at $t = 1$, 1 sends to 10; at $t = 2$, 10 sends to 100; at $t = 3$, 100 sends to 1000, etc. Thus all messages passing through 0 arrive at their proper destination one time unit earlier than before (when $n = 2^m$), and since the recipient is the same individual who by (A7) would send to 0 at the next time, all the information previously passing through 0 is passed to the entire group without loss. If $n = 2^m-2$ we omit the next binary number which never sends directly to 0, i.e. 11 in binary notation and use the same modified rule for sending messages which by (A7) would go to 11. This can be continued for all n between 2^{m-1} and 2^m, the numbers successively omitted being 0, 11, 101, 110, 1001, 1010, 1100, 1111, etc.; each of these is the smallest number which differs by two digits from all the preceding ones. This proves (A1) for $r = 1$.

For $r > 1$, the procedure is similar using $(r + 1)$-ary numbers. If $n = (r + 1)^m$

$$i \text{ is written as } e_{im}\, e_{i,m-1} \cdots e_{i2}\, e_{i1} \tag{A8}$$

where each e is one of the integers $0, 1, 2, \ldots, r$ and (A8) is interpreted as

$$i = e_{i1} + (r + 1)\, e_{i2} + (r + 1)^2\, e_{i3} + \ldots + (r + 1)^{m-1}\, e_{im} \quad . \tag{A9}$$

The sending instructions are the same as (A7) but now there are r numbers satisfying (A7). The reasoning showing that a message will pass between every pair in at least m steps is similar to that for $r = 1$.

For $n = (r + 1)^m-1$ we omit the number 0, and modify the sending so that any message for 0 according to (A7) goes at time 1 to any one of the number 10, 20, ..., r0 (in (r+1)-ary notation). It does not matter which of these is selected because each sends to all the others at time 2. Similarly at time 2, messages from 10, 20, ..., r0 directed to 0 by (A7) are now sent instead to any one of the numbers 100, 200, ..., r00; these messages contain the information directed to 0 at time 1, and are included in the messages sent to each of the numbers 100, 200, ..., r00 at time 3 by (A7). Thus no information is lost.

The numbers to be omitted for successively smaller values of n are 0, 1, 2, ..., r-1. Next we omit in succession 10, 20, ..., (r-1) 0 but retain r0. The next numbers to be omitted are 11, 21, ..., (r-1) 1 but r1 is retained. The scheme is that for each number omitted we retain at least one of its recipients at each t according to (A7). In this way any message directed to a missing number by (A7) has a destination when the next digit is looked at in accordance with the modified instructions. All messages get passed on with no loss of information. It is also easy to see that the rules about which numbers must be retained allow us to reach any n between $(r + 1)^{m-1}$ and $(r + 1)^{m}$.

When n is not a power of $r + 1$ the sending pattern is not unique, the rules above being just one possible scheme.

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