COWLES COMMISSION DISCUSSION PAPER: ECONOMICS NO. 2102

Efficient Communication Networks

D. Bratton

April 29, 1954

§ 1 - Introduction

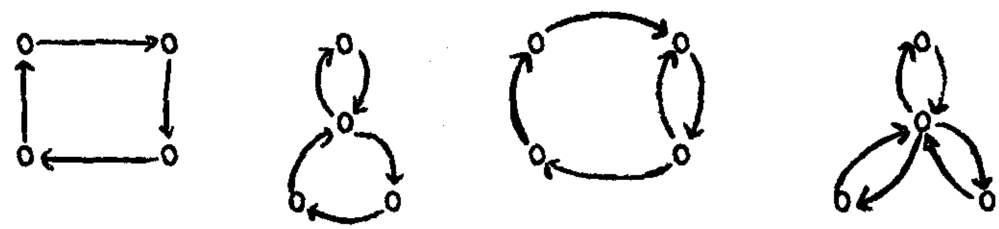
A communication network for a set P of people provides links between certain couples of elements of P. These links are uni-directional. Information can be sent from a person x to a person y providing there is a path from x to y in the network. The number of links in the path is called the length of the path. A path is called a route when there is no shorter path with the same terminals.

A network is called adequate when each person can reach each other person. One then is concerned with the length of the longest routes; this is the maximum number of steps which a message must make; it is called the solution time (& denoted T) of the network.

We are here concerned with the adequate networks which are efficient for the value of T and the number ℓ of links employed. More generally, the problem which we study is, what is the smallest value of T attainable for a given number p of people using an adequate network with ℓ links?

This is a special case of the problem of the cheapest network formulated by Radner & Tritter in CCDP: Economics No. 2098.

To clarify ideas, the following diagrams present four adequate networks for 4 people:



$\ell = 4$	$\ell = 5$	$\ell = 5$	$\ell = 6$
$T = 3$	$T = 3$	$T = 3$	$T = 2$

§ 2 - A characteristic number associated with a network

The first of the above illustrations is an example of a cyclic network; it is the unique adequate network where $\ell = p$. In general $l \geq p$. The number $c = 1 - p + \ell \geq 1$ is an important number associated with a network: it is the basic number of cycles in the topological space of the network, and is a topological invariant. (You can see that interpolating an additional person at the middle of a link increases both p and ℓ by 1, and hence leaves c invariant.) As the range of ℓ is $p \leq \ell \leq p(p-1)$, one has $1 \leq c \leq (p-1)^2$ At the lower extreme $c = 1$ we get the unique cyclic network; at the upper, the unique network (it could be called the "saturated" network) where each pair of persons is linked in both directions. The solution time for $c = 1$ is $T = p - 1$; for $c = (p-1)^2$; $T = 1$. For $1 < c < (p-1)^2$ one has $p - 1 > T > 1$, clearly.

Indeed, for $c = p - 1$ there is available the "centralized" network illustrated by the last of the above four illustrations, for which always $T = 2$. It is easy to see that for $p - 1 \leq c < (p-1)^2$, 2 is the least possible value of T for any network.

Our interest thus lies in the values $1 < c < p$. And here we have necessarily $1 < T < p$ also.

The network R being adequate, each person x must have at least one link leading to him ("entrance link") and one link leaving him ("exit link"); x is called a <u>node</u> if there are additional links.

A path each of whose extremities is a node and such that no other nodes occur in it is called a <u>branch</u>. b denoting the number of branches, n the number of nodes, one has $c = 1 - n + b$ (because of the invariance of c).

As any two of the numbers c, ℓ, p determine the third, we will specify them at various times by specifying only two of them.

<u>§ 3 - Foils - the one-node networks</u>

When $c > 1$, there is at least one surplus link: if we associate to each person one of his exit links, there is thus left over $c - 1 \geq 1$ links; choosing one of them, the person at its beginning is thus found to have two exits -- he is a node.

A network R with exactly one node has a very simple structure: the node is at the center, from which the other persons are arranged along branches which begin and end at the mode: the graph is a number of "leaves" attached to a central point. The number of these leaves is exactly c. Let us call such a network a "foil".

Thus, in order that an adequate network R have just one node it is necessary that $1 < c < p$. Conversely for each value of c and p such that this inequality holds, there exists a foil which realizes them; it has c leaves.

The solution time of a foil is particularly easy to compute. Define the function σ of a real variable as follows: $\sigma(x + 1) = \sigma(x) + 1$; $\sigma(0) = 0$, $\sigma(x) = \frac{1}{2}$ for $0 < x < 1$. For each integer n, $\sigma(n) = n$; it is a step-function. The least solution time for foils with given ℓ and c is

$$T_F = 2\,\sigma\left(\frac{\ell-1}{c}\right) - 1$$

It is obtained by dealing out the people around the leaves, and is unique up to a permutation of the people. We thus shall refer to it as <u>the foil</u> corresponding to the numbers p and ℓ.

Using the properties of the function σ one obtains rational estimations of its value, and hence of T_F:

$$\frac{2\ell}{c} - 2 \leq T_F \leq \frac{2\ell - 4}{c} \qquad (1 < c < p)$$

These bounds allow a leeway of at most two consecutive values of T; indeed one can see that T_F is precisely the least integer $\geq \frac{2\ell}{c} - 2$, <u>except</u> when the remainder upon dividing ℓ by c is ≥ 2 and $\leq \frac{1}{2}c$.

<u>§ 4 - Search for fast networks</u>

It is easy to believe that the foil is the quickest adequate network. While this is probably true for $p \leq 14$, a counterexample exists for $p = 15$. But the exceptions would at least seem to be scarce.

In any case, the formula $T_F = 2\sigma\left(\frac{\ell-1}{c}\right) - 1$ offers an upper bound on the least time for any network. How good is it? Perhaps it is asymptotic (as $p \to \infty$) to the true value? (We shall see more about this later.)

We thus begin a search for quicker networks, which justifies the following classification: let us call a network R <u>fast</u> if its solution time is less than that of all the corresponding foils; otherwise <u>slow</u>. (Assuming throughout that $1 < c < p$.)

What are the necessary properties of fast networks?

<u>A fast network has three or more nodes</u>.

To see this, let R be a 2-node network, A, B its nodes. There exists a branch going from A to B and vice versa. There may be some leaves attached at A or B. One obtains a net network R' by the following transformation. Any leaves at B are moved to A. While retaining the shortest branch a from A to B and the shortest branch b from B to A, remove all the other branches between the two nodes at the point where they touch B and attach this end to A. It is easy to see that $T' \leq T$, R' has one node, and the same p and $\not\!\!1$.

<u>If in a network R there is a node with a unique entrance (resp. exit), then there exists a network R' with one less node, same p and $\not\!\!1$, and such that $T' \leq T$.</u>

Case of unique entrance: A node B has only one entrance; the unique branch a entering B cannot be a leaf, so leaves from a node $A \neq B$. One obtains R' by leaving intact a route from B to A but moving to A all the tails of the remaining exit branches at B.

We shall show in detail that $T' \leq T$. For this, let r' be a longest route in R', supposed, in order to argue a contradiction, unmatchable by a route in R. r' must then use one of the altered links--one of the newly created exits at A, -- and hence $A \in r'$. Supose $B \in r'$: it would have to precede A (for if B follows A, then the unique path a from A to B belongs to r', hence r' visits A twice!). But then as R' has unique path p from B to A, r' = aBpAu, where u is a route in R' from A to some point. But the path r = aBpAu <u>in R</u> is also a route of the same length, contradicting the definition of r'.

On the other hand, suppose $B \notin r'$. One defines a path r in R, r = (part of r' preceding A) AaB (part of r' succeeding A). r is longer than r', which again contradicts the definition of r'. q.e.d.

If in a network R each branch is of length ≥ 2, then each route of R can be extended until it terminates on non-nodes (clearly); its length depends only on the length of these terminal branches together with a node-

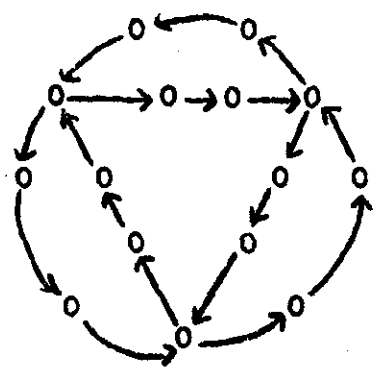
node route. Curiously enough, this holds also for any network where each node has at least two entrances and two exits; one has here:

$$T = \max_{u \neq v} [-2 + \ell(u) + l(v) + d(e(u), b(v))]$$

where u and v run over the branches of R. (Notation: $\ell(u)$ is the length of u, d(A,B) is the distance from the point A to the point B, e(u) (resp. b(u)) the end (resp. the beginning) of the branch u.)

<u>§ 5 - A fast 3-node network</u>

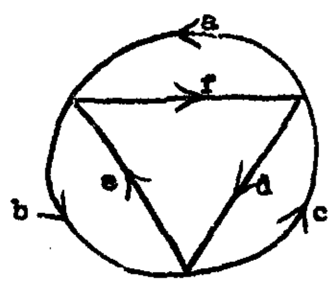
A



The network A at the left is fast, $p = 15$, $\ell = 18$, $c = 4$, and the solution time T is 7, one less than T_F. It is interesting to observe that $\frac{2\ell}{c} - 2 = T < T_F$.

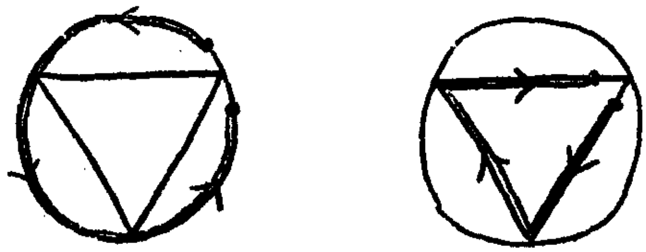
One might hope to find other fast networks by generalizing A. The first thing to do is to consider any network A' with the same oriented graph as A but with an arbitrary number of people on each of the six branches. The lengths of the branches must be ≥ 2 or else $c > p$.

A'



<u>For the generalized networks A' one always has $T \geq \frac{2\ell}{c} - 2$.</u>

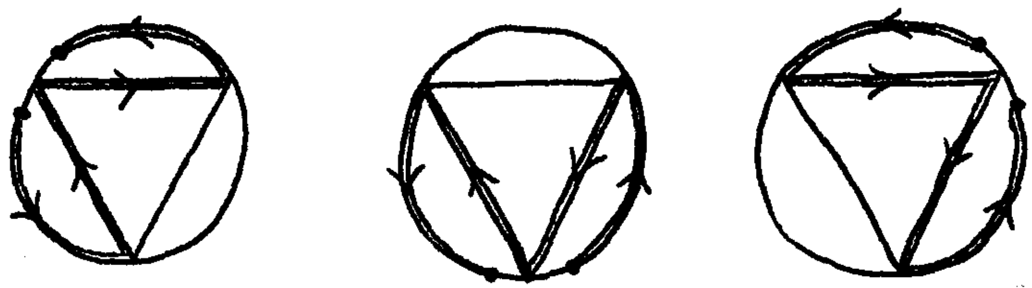
The proof of this distinguishes two cases. Case I: at least one of the outside branches, and at least one of the inside branches, is a route between its nodes. Here we can always find two routes r_1, r_2 as indicated in the following figures



By their construction, $\ell(r_1) + \ell(r_2) = \ell - 4$, hence

$$t \geq \max_i l(r_i) \geq \frac{1}{2} [\ell(r_1) + \ell(r_r)] = \frac{1}{2} (\ell-4) = \frac{2\ell}{c} - 2.$$

Case II: None of the outside branches is a route between its nodes. Here we can always find three routes r_1, r_2, r_3 as indicated in the following figures:



Therefore $T \geq \max_i \ell(r_i) \geq \frac{1}{3} \Sigma \ell(r_i) = \frac{1}{3} (2\ell - 6) > \frac{\ell}{2} - 2 = \frac{2\ell}{c} - 2.$
q.e.d.

We pause here to remark,

<u>If in a network</u> R <u>where</u> $1 < c < p$ <u>one has</u> $T \geq \frac{2\ell}{c} - 2$, <u>then</u> $T \geq T_p - 1$.

This results from the estimate of the time T_p of the corresponding foil, given in § 3.

Continuing our analysis of the networks A', we ask, does T ever equal $T_p - 1$? This could only occur under the conditions of case I of the above proof, howeover, by § 3, $\ell \equiv 2 \pmod 4$, so that $\frac{\ell}{2} - 2$ is an integer, and hence $T = \frac{\ell}{2} - 2$. This equality, in the light of the relations exhibited in case I, implies that $T = \ell(r_1) = \ell(r_2) = \frac{\ell}{2} - 2$; i.e., $a + b + c = d + e + f = T + 2$.

<u>It follows that each branch of A' is a route between its nodes.</u>

To see this, suppose that the branch b of A' is not a route. One can then see a route r_1 in A' of length $(a-1) + f + d + (c - 1)$, and a route r_2 of length $(b-1) + (e - 1)$.

Hence $T \geq \frac{1}{2} [\ell(p_1) + \ell(p_2)] = \frac{1}{2} [a + b + c + d + e + f - 4]$ $= \frac{1}{2} [2 (T + 2) - 4] = T$. Hence $T = \ell(p_1) = \ell(p_2)$, i.e., $b + e - 2 = T$, so $b = e + f$, contradicting the hypothesis on b.

<u>All the branches of A' are of equal length.</u>

To see this, it suffices to show that $a \leq e$, $b \leq d$, $c \leq f$. By a suitable selection of routes (keeping in mind that each branch is a route), one obtains the inequalities

$$a + f + d - 2 \leq T$$

$$b + e + f - 2 \leq T$$

$$c + d + e - 2 \leq T$$

which imply the desired inequalities, as $T = d + e + f - 2$.

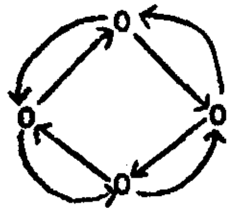
We have demonstrated the following:

<u>In order that a network A' with the graph of figure 2 be fast, it is necessary and sufficient</u> that each branch <u>of A' have the same length</u> a, where a is an odd number ≥ 3. <u>Its solution time T is then equal to T_F $- 1 \geq \frac{2\ell}{c} - 2$.</u>

There are additional results, which will be quoted briefly. Adding leaves to A' makes it slow; as does removing any of its branches or adding duplicate ones: the A' are the only fast 3-node networks.

If one attempts to generalize A' by taking an arbitrary number of nodes and connecting them all 2 - by - 2, each branch being of the same length, the resulting network is not fast unless $n = 3$.

If on the other hand one connects up n nodes in a "double circle" the result is again slow,



except for $n = 3$.

§ 6 - The conjecture $T \geq \frac{2\ell}{c} - 2$

The study of the class of networks A' of the previous § inspires several conjectures which will occupy most of the remainder of this paper. For one thing, the lower bound $\frac{2\ell}{c} - 2$ for the time of the foils also held for all the networks A'. We found in turn that any network where T exceeds this quantity is at best one unit faster than the foils. Is it generally true that $T \geq \frac{2\ell}{c} - 2$? $(1 < c < p)$. This would indeed be a very satisfying result. I have not been able to settle this question, but can report a few skirmishes which may be of some interest.

(1) Given a network R, suppose that we can find two subnetworks R_1, R_2 satisfying the following conditions:

(a) $T \geq \max(T_1, T_2)$

(b) $c - c_1 - c_2 = \ell - \ell_1 - \ell_2 \geq 0$

Then $\frac{\ell}{c} \leq \max\left(\frac{\ell_1}{c_1}, \frac{\ell_2}{c_2}\right)$.

Thus if $T_i \geq \frac{2\ell_i}{c_i} - 2$, then also $T \geq \frac{2\ell}{c} - 2$.

This is a purely arithmetical result: one has $\frac{\ell}{c} = \frac{\ell_1 + \ell_2 + k}{c_1 + c_2 + k} =$

$\frac{c_1 a_1 + c_2 a_2 + k}{c_1 + c_2 + k}$ (where $a_i = \frac{\ell_i}{c_i} \geq 1$).

The latter is a convex combination of a_1, a_2, and 1, and therefore its value is $\leq \max a_i$.

This method is easily generalized to the case where there are n subnetworks R_i such that $T \geq \max T_i$, concluding that

$$\frac{l - b}{c - a} \leq \max_i \frac{l_i - b}{c_i - a}$$

providing that the constants a and b satisfy the conditions

(A) $(n - 1)(a - 1) + q - k \geq 0$

(B) $(n - 1) b - k \leq [(n - 1)(a - 1) + q - k] \dfrac{l_i - b}{c_i - a}$

where $q = - p + \Sigma p_i$, $k = - l + \Sigma l_i$.

However, it is not clear to me how to go about finding the subnetworks R_i.

(2) Another possiblity is to alter a given network R to form another adequate network R' where $\Delta c < 0$ and $\frac{\Delta l}{l - a} \geq \frac{\Delta c}{c}$, a being some chosen number. If by induction on c one can conclude that $T' \geq 2 \frac{l' - a}{c'} - b$, then also $T \geq 2 \frac{l - a}{c} - b$.

For example, suppose that R is a network where $T < \frac{2l}{c} - 2$ and where the routes are not unique. This says that there exists two nodes A, B with two different routes r_1, r_2 from A to B. One obtains a new network R' by "collapsing" the two routes r_1, r_2 onto each other so that their respective people and links merge two-by-two. We have lost one cycle, so $c' = c - 1$. Also $T' \leq T$ and $l' = l - d(A,B)$. By induction on c, we can assume that $T' \geq 2 \frac{l'}{c'} - 2$. Hence

$$\frac{l}{c} > \frac{T + 2}{2} \geq \frac{T' + 2}{2} \geq \frac{l'}{c'} = \frac{l - d(A,B)}{c - 2} \quad ;$$

hence $d(A,B) > \frac{l}{c}$ in R.

Is it possible to deduce a contradiction from this? If so, we would reduce the conjecture to the case of unique routes.

(3) A reduction to the case where there is at most one branch which is not a route is possible.

For if there are two such branches in R, of lengths a, b, upon removing a we get a new network R' where $c' = c - 1$, $T' \leq T$. By induction on c, we can assume that

$$\frac{\ell}{c} > \frac{T+2}{2} \geq \frac{T'+2}{2} \geq \frac{\ell'}{c'} = \frac{\ell - a}{c-1} .$$

Hence $a > \frac{\ell}{c}$, and similarly $b > \frac{\ell}{c}$. But there exists in R a route beginning at the one branch and ending at the other, of length $\geq a + b - 2 > 2\frac{\ell}{c} - 2$, a contradiction.

§ 7 - A combinatorial problem

There is another clue lying in the study of the fast networks of § 6 to be used to generate a conjecture. The proofs there proceeded by choosing a set of routes which covered the network. In each case we found two routes which jointly used up all the people in the network. We could have just as well picked 4 routes so that each person is used at least twice.

Covering conjecture - R being an adequate network with $c \geq 2$, there exists a set of c routes such that each person in R occurs in at least two of them.

This condition has held up admirably on all the particular networks that I am able to sketch on a piece of paper. It is obvious for foils. It clearly holds when $c = 2$. I have convinced myself that it holds for all three-node networks. It is appealing in that it is purely combinatorial in nature, making no reference to the lengths of routes, etc. Its value for us lies in this; that

One deduces from the covering conjecture that $T \geq \frac{2p}{c} - 1.$

Otherwise said, $T \geq \frac{2\ell + 2}{c} - 3$. For, n_i denoting the number of people on the i^{th} route $r_i (1 \leq i \leq c)$, one has $\Sigma n_i \geq 2p$. Hence $1 + T \geq \max [1 + \ell(p_i)] \geq \frac{1}{c} \Sigma [1 + \ell(p_i)] = \frac{1}{c} \Sigma n_i \geq \frac{2p}{c}$.

This lower estimate for T is within one unit of the estimate of § 7. If each branch of R has length ≥ 2 one can actually strengthen it so that it coincides with the earlier one.

Well, what openings are there for proofs of the covering conjecture? The thought that plagues me is that, c being the basic number of cycles in the topological space of the network, one should be able to find a route corresponding to each such cycle.

Indeed, in the foils and in the fast networks so far discovered, there are always c cycles which are actually circular paths in the network. Is this generally true?

The simplest scheme for a proof that occurs to me is to prove it separately for c even and c odd and in either case to do so inductively by building up R by adding two branches at a time. This step increases c by 2 so that we can introduce two new routes. We can choose two routes so that we jointly cover each of the new branches twice. But when we go into this in detail, we get into trouble over the unknown effect on previous routes produced by adding new branches.

One might use induction backwards, proving the conjecture in the limit case where there are a given number k of branches between every couple of nodes, and then work backwards by removing two branches at a time. Similar trouble occurs.

The only step which is perfectly clear is this: if R has two branches u,v which are not routes between their nodes, then a covering of the smaller network R' where these branches are removed becomes a covering of R when supplemented by a route from a point of u to a point of v, and by a second route in the opposite direction. This reduces the problem to the networks where there is at most one "redundant" branch.

As leaves are always "redundant" in the above sense, we need never consider networks with 2 or more leaves.

(Heuristically, this suggests that the fast networks will abandon leaves altogether. Is this not a likely conjecture?)

Maybe we are being too highbrow about all this: there may be a direct method of constructing the desired set of c routes. In this case we would only have to check the validity of the construction rather than prove an existence theorem.

Finally, we ask, is there a realistic interpretation of the covering conjecture?

§ 8 - Conclusion

Perhaps the only escape from the position where we were stranded in the previous section is by making a deeper study of the nature of routes in a network. Several possibilities exist which are being studied at the present time. For one thing, one can work out the characteristic properties of the distance function d(A,B). (This leads to a more general notion of network which has been studied by A. Shimbel and will shortly appear in published form.)