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On the Optimal Path of Expansion of a Firm

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On the Optimal Path of Expansion of a Firm

1. Consider a firm which produces a single commodity for immediate sale and has a fixed profit margin. Its output is limited by the size of plant, which we assume to be capable of expansion at any non-negative rate at a cost proportional to the capacity added. Given the levels of demand over a future period of finite length T, what are the rates of investment that result in the maximization over the period T of the profits discounted at the market rate of interest?

2. Let $x(t)$ $\quad 0 \leqq t \leqq T$

denote the (fixed) levels of demand,

$y(t)$ the (unknown) capacity of plant in terms of possible output,

c the ratio of the cost of expansion per unit constructed to the profit per unit of output,

r the rate of interest.

Then the current profit from sales is proportionate to Min (x,y). Net profit after expense for investment over the whole period is

$$p\,[y] = \int_0^T [\text{Min}\,(x,y) - c\dot{y}]\, e^{-rt}\, dt,$$

where $\dot{y}$ denotes the derivative of y with respect to time t.

3. The problem is to maximize $p[y]$ subject to $\dot{y} \geqq 0$. The peculiar features of this calculus of variations problem are that the integral does not have a continuous derivative and that the unknown function must be monotonically non-decreasing. Let us introduce a function

$$\varphi(z) = \begin{Bmatrix} 1 \\ u \\ 0 \end{Bmatrix} \text{ if } z \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \qquad \text{where } 0 \leqq u \leqq 1 .$$

Now since y is non-decreasing $\int_0^T \text{Min}\,(x,y)\, e^{-rt}\, dt$ may be written

$$\int_0^T \dot{y} \left[\int_t^T \varphi\,(x(s) - y(t))\, e^{-rs}\, ds \right] dt$$

provided $y(0) = 0$ as we may assume without restriction. This formula is obtained by summation of horizontal strips rather than vertical ones. Writing for shortness

$$\int_t^T \varphi\,[x(s) - y(t)]\, e^{-rs}\, ds = g(y(t), t)$$

the maximand becomes

$$\int_0^T \dot{y}\, [g(y(t),t) - ce^{-rt}]\, dt .$$

4. In order to take account of the fact that $\dot{y} \geqq 0$ the Euler equations must be derived in a special form as follows. Consider a variation of

$$\int_0^T f(y,\dot{y},t)\, dt \qquad \text{by } \epsilon\,\eta \text{ where } \eta(0) = 0,\ \dot{\eta}(t) \geqq 0 \text{ if } \dot{y}(t) = 0$$

$$\frac{d}{d\epsilon} \int_0^T f(y + \epsilon\eta,\ \dot{y} + \epsilon\dot{\eta},\ t)\, dt$$

$$= \int_0^T [f_y \eta + f_{\dot{y}}\, \dot{\eta}]\, dt = \int_0^T \dot{\eta}\, [f_{\dot{y}} + \int_t^T f_y\, ds]\, dt .$$

The last expression is obtained by partial integration, the term $\eta \int_t^T f_y\, dt$ having vanished at $t = 0$ and $t = T$. A maximum requires that

$$f_{\dot{y}} + \int_t^T f_y\, dt \begin{Bmatrix} = \\ \leqq \end{Bmatrix} 0 \quad \text{where } \dot{y} \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

5. Applying this to $f = \dot{y}\, [g(y,t) - ce^{-rt}]$ we have

$$g - ce^{-rt} + \int_t^T \dot{y}\, \frac{\partial g}{\partial y}\, dt \begin{Bmatrix} = \\ \leqq \end{Bmatrix} 0 \quad \text{if } \dot{y} \begin{Bmatrix} > \\ = \end{Bmatrix} 0.$$

Since $\dfrac{\partial g}{dt} = \dfrac{\partial g}{\partial y}\, \dot{y} + \dfrac{\partial g}{\partial t}$

$$\int_t^T \dot{y} \frac{\partial g}{\partial y} \, dt = g(T) - g(t) - \int_t^T \frac{\partial g}{\partial t} \, dt$$

But $$g(T) = \int_T^T \varphi \, e^{-rt} \, dt = 0$$

$$\frac{\partial g}{\partial t} = \varphi(x(t) - y(t)) \, e^{-rt} .$$

Thus finally

$$\int_t^T \varphi(x(t) - y(t)) \, e^{-rt} \, dt \begin{Bmatrix} = \\ \leqq \end{Bmatrix} ce^{-rt} \quad \text{if} \quad \dot{y} \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

6. In particular if $\dot{y} > 0$ one has by differentiation

$$\varphi(x(t) - y(t)) = rc$$

We conclude that expansion can never pay if $rc > 1$, a fact that may be verified also directly since $\frac{1}{r}$ is the return from a unit investment over an unlimited period if the capacity is always utilized. If T is finite, as assumed here, $rc < 1$. Remembering the definition of φ we conclude that whenever expansion takes place $(\dot{y} > 0)$, it follows precisely the sales curve.

From the second last relation we obtain that expansion does not take place at times when the discounted future return from the marginal layer of the capacity set falls short of the cost of the investment.