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A Model for Stochastic Decision Making[1/]

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Denote: u_i = a stochastic "utility variable" assigned to a_i

$\hat{u}_A$ = Max u_i, $a_i \in A$, $\hat{u}_{A-a} = \underset{j \neq i}{\text{Max}}\ u_j$

a_c = alternative chosen

A = the set of possible choices.

(*Axiom) $$P_{i,A} = P(a_c = a_i) \overset{*}{=} P(\hat{u}_A = u_i) = \int_{x=-\infty}^{\infty} P(\hat{u}_{A-a_i} \leq x \quad x < u_i \leqq x + dx)$$

$$P_{A'_i A} = \sum_{a_i \in A'} P_{i;A}\,, \quad P(\ \ ) = \text{Probability of } (\ \ ).$$

The conditional probability that $a_c = a_i$ when $a_c \in A' \subset A$, $a_i \in A'$ we denote

$$P\,(a_c = a_i \mid a_c \in A') = \frac{P_{i;A}}{P_{A';A}} = P_{i;A',A}$$

If $P_{i;A',A}$ is independent of $A - A' = A''$ we say that the probabilities of choice $P_{i;A'}$ inside a group of alternatives is independent of alternatives

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outside the group. In this case

$$(1) \qquad P_{i;A',A} = \frac{P_{i;A}}{P_{A';A}} = \frac{P_i}{P_{A'}} = P_{i;A'} = \int_{x=-\infty}^{\infty} P(u_{A'-a} \leqq x, x < u_i \leq x + dx)$$

Definition.

If the condition (1) is fulfilled for all relevent subsets A' of A we say that the probabilities of choice are relatively stable to non-admissible alternatives $A - A'$ and call the probability field describing the choice situation as a relatively stable probability field of choice.

We will first study the case when A' contains only three alternatives $A' = (a_1, a_2, a_3)$ and denote $P_{ij} = P_{i;A'-a_k}$ where (i,j,k) is some permutation of $(1,2,3)$. If further

$$P_{ijk} = P(\hat{u}_{A'} = u_i, \hat{u}_{A'-a_i} = u_j)$$

we have always

$$P_{ij;A'} = P_{i;A'} + P_{kij} , \; P_{ijk} = P_{jk;\bar{A}'} P_{j;A'}$$

If we consider a relatively stable probability field of choice we have

$$P_{ij;A'} = \frac{P_i}{P_i+P_j} , \quad P_{ji;A'} = 1 - P_{ij;A'} = \frac{P_j}{P_i+P_j} ;$$

$$P_{ijk} = \frac{P_i P_j}{P_j+P_k} = P_i P_{jk;A'} = \frac{P_i P_j}{1-P_i} ; \quad \text{if} \quad P_i + P_j + P_k = 1$$

If $P_i \geqq P_j \geqq P_k$ we have

$$\frac{P_{ij}}{P_{ji}} = \frac{P_i}{P_j} \geqq 1 \quad \frac{P_{jk}}{P_{kj}} = \frac{P_j}{P_k} \geqq 1 \quad \frac{P_{ik}}{P_{ki}} = \frac{P_i}{P_k} = \frac{P_{ij}}{P_{ji}} \cdot \frac{P_{jk}}{P_{kj}} \geqq 1$$

If thus $P_{ij} \geq \frac{1}{2}$ and $P_{jk} \geq \frac{1}{2}$ then the transivity property $P_{ik} \geqq \frac{1}{2}$ holds. For not relatively stable probability fields of choice it can however happen that the transitivity relation in this sense is not fulfilled.

(Ex.* $P_{123} = 0.19$, $P_{231} = 0.28$, $P_{132} = 0.24$, $P_{213} = 0.05$, $P_{321} = P_{312} = 0.12$

$P_{12} = P_{123} + P_{132} + P_{312} = 0.55$, $P_{23} = 0.52$ but $P_{13} = 0.48$

$P_1 = 0.43$, $P_2 = 0.33$ $P_3 = 0.24$,

the probability $P_{213} = 0.05$ is however peculiarly small compared with $P_{231} = 0.28$. If the transitivity property however is defined by help of $P_i \geq P_j \geq P_k$ it is always possible to partition a set of alternatives A into three sets $A_i^{(+)}$, $A_i^{(0)}$, $A_i^{(-)}$ such that the alternatives belonging to $A_i^{(-)}$ can be said to be inferior to a_i the alternatives belonging to $A_i^{(0)}$ equivalent to a_i and the alternatives $A^{(+)}$ as superior to a_i. Thereafter it is in principle possible to apply the method of indifference surfaces $A_i^{(0)}$ to the theory of choice. A multivariate normally distributed probability field for u_i always fulfil the transitivity relations in the first sense; We have namely in this case

$$P_{ij} = \Phi\left(\frac{m_i - m_j}{\sigma(u_i - u_j)}\right) \geq \frac{1}{2} \text{ if } m_i \geq m_j$$

$$P_{jk} = \Phi\left(\frac{m_j - m_k}{\sigma(u_j - u_k)}\right) \geq \frac{1}{2} \text{ if } m_j \geq m_k.$$

$$P_{ik} = \Phi\left(\frac{m_k - m_k}{\sigma(u_i - u_k)}\right) \geq \frac{1}{2} \text{ if } P_{ij} \geq \frac{1}{2}, \text{ and } P_{jk} \geq \frac{1}{2}.$$

It is however not possible to conclude that $P_{ik} \geq \text{Min}\,(P_{ij}, P_{jk})$ because $\sigma(u_i - u_k)$ can be much larger than the other standard deviation.

It is perhaps of interest not note that for a relatively stable probability field of choice

$$P_{ijk} = P_i P_{jk} = \frac{P_{ij} P_{jk} P_{ik}}{1 - P_{ji} P_{ki}}$$

$$P_i = \frac{P_{ij} P_{ik}}{1 - P_{ji} P_{ki}}$$

* When P_{ij} are given for all pairs (i,j) we have two degrees of freedoms left for the six probabilities of the kind P_{ijk}, which can be used for fixing P_{321} and P_{312}.

If $P(\hat{u}_{A'-a_i} \leq x,\ x \leq u_i \leq x + dx) = \frac{P_i}{P_{A'}} d_x P(\hat{u}_{A'} \leq x)$

for every A' the probability field of choice is surely relatively stable. In this case all the variables u_i are independent with marginal distributions $(P(u_A \leq x))^{P_i}$.

For relatively stable probability fields of choice the "utility distance" between two alternatives can be defined by means of the formula:

$$d_{ij} = \frac{\log P_{i;A'} - \log P_{j;A'}}{\alpha} = \frac{\log P_i - \log P_j}{\alpha} .$$

In this case we have

$$P_{i;A} = \frac{e^{\alpha d_{ij}}}{\sum\limits_{a_i \in A} e^{\alpha d_{ij}}} , \quad P_{i;A'} = \frac{e^{\alpha d_{ij}}}{\sum\limits_{a_i \in A'} e^{\alpha d_{ij}}}$$

If

$$P(\underset{a_i \leq A}{u_i - u_j \leq x_{ij}}) = 2^{-\sum\limits_i e^{\alpha(d_{ij} - x_{ij})}}$$

the probability field of choice is relatively stable with $P_{i;A'}$ given by the formulas mentioned. The number d_{ij} denote the median of $u_i - u_j$ and α determines the concentration of probability in the neighborhood of this point. If $\alpha \to \infty$ all $P_{iA'}$ for which d_{ij} is smaller than $\max\limits_{a_i \in A'} d_{ij}$ converges to zero.

In this case of relatively stable probability field of choice we have

$$(2) \qquad P_{i_1 i_2 \ldots i_u} = \prod_{v=1}^{u} P_{i_v;B_v} = \prod_{v=1}^{u} \frac{P_{i_v}}{P_{B_v}}$$

where

$$P_{B_v} = \sum_{\mu=v}^{u} P_{i_\mu} = \sum_{\mu=1}^{u} P_{i_\mu} - \sum_{\mu<v} P_{i_\mu} \leq 1 - \sum_{\mu<v} P_{i_\mu}$$

The probability $P_{12\ldots n}$ is the largest of all $P_{i_1\ldots i_u}$ if $P_1 \geq P_2 \ldots \geq P_u$ denote the n largest P_i among P_i ; $a_i \in A$.

A probability field of choice with the property (2) could be called a relatively independent-stable probability field of choice.

A quite different espect on the decision making problem we have when we study the problem of estimating x_i, when to different alternatives a_i corresponds different probability distributions over possible consequences of the choices made hypothetically. But we shall not here try to work out the connections between $P_{i_1\ldots i_n}$ and the forms of these distributions.