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On the Theory of Location in the Short Run

by

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> Das eigentliche Geschaeft der Oekonomen ist nicht die miserable Wirklichkeit suerklaeren sondern sie zu verbessern. Die frage nach dem besten Standort ist ungleich wuerdiger als die Feststellung des tatsaechlichen.
>
> -- A. Loesch, "Die räumliche Ordnung der Wirtschaft"

Introduction.

The theoretical study of locational choices as investment decisions and of their long-run implications for the location pattern of industries must cope with deep difficulties in the analysis of internal economies of scale and of external economies of industrial localization and general economic concentration. In the short run, on the other hand, we face rigidities such as the capacities of plants, and limitations on the rate of supply from various resource deposits, and on the quantities of product that can be disposed of in each market.

Existing location theories do not distinguish explicitly 3/ between the long and short run. Among the more ambitious attempts to reduce the whole of location phenomena to one simple scheme is the recent revival (Isard [5]-[8]) of earlier substitution analysis (Predoehl[13], Furlan [3]) in

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3/ In what must be regarded as the most systematic exploration of location economics, Loesch's "Die räumliche Ordnung der Wirtschaft" (The Spatial Structure of the Economy [12]), the distinction between long and short run is not drawn.

more rigorous form. All locational decisions are derived according to the principle of substitution in which transportation outlays, called "distance inputs" are treated in the same way as other production inputs. The important admission is made, however, that analysis of concrete situations may become complicated (Isard, [5] p. 396). This would be the case unfortunately for "substitution between distance inputs and between various outlays and revenues" (p. 397) such as are incurred when sources of different f.o.b. price and limited capacity are available for the supply of material inputs, or when markets are to be allocated to plants of limited capacity. Another serious shortcoming seems to be the reliance on differentiability (smoothness) of the cost or profit functions, and the fact that the substitution principle assures us merely of maximization in the small. Only in special cases[1] are the first and second order conditions which have been derived sufficient for a maximum in the large.

The successful application in recent years of activity analysis to transportation problems (Koopmans [9], [10], Hitchcock [4], Samuelson [15]) suggests that an alternative approach to the short-run problems of location remains to be tried. A preliminary attempt is made here on two levels, the level of the firm operating branch plants (Part I) and of an economy composed of regions (Part II); a few remarks are inserted on the competitive industry (Section 2). The problems of location in the long run are here left untouched. Their study by means of activity analysis has to await the successful treatment of allocation problems involving indivisibilities of processes and commodities.

[1] When the profit function is convex.

I. The Allocation Problem for a Firm

1. The Model.

1.1 Let a firm which produces a single commodity in a number of branch plants be deciding on its production and sales program. We assume that the technology is such that apart from fixed plant all inputs are in constant proportion to output and that their ratio is independent of the location. To begin with let the price of inputs and outputs be the same at all locations.

Then the profitability of production at any location depends on transportation cost only. There exists a critical value of transportation cost that renders production at a location just profitable. If transportation cost is less than this part of the revenue may be regarded as "rent" on plant. Rent per unit of output is then a measure of the profitability of a plant location.

Nothing can be said under these simple conditions about the scale of operations. Only a simple question can be answered: where is production definitely unprofitable and where might it be profitable?

So long as prices of resources are the same at all "resource deposits" there is no problem of choice among the sources of supply. If transport cost is proportional to distance it is always the geographically nearest location where an input can be obtained at lowest cost; and if market prices are the same, each plant delivers to the nearest market location.

1.2 Now let product prices to the firm be different at different markets and resource prices different at different deposits. Moreover, let transport cost per unit distance vary with distance shipped. Then the resource deposit which is nearest for any production location is replaced by the one which is "cheapest" -- the one for which the resource can be brought to the production site for the least total cost per unit; and the nearest market is replaced by the "most lucrative" market -- the one for which the net revenue (net of transport cost) from the sale of a unit product is the largest.

1.3 So far we have not made precise the allocation problem of the firm. We now specify that its goal is profit maximization.

1.4 Suppose that the firm makes the decision about what plants to run and is faced with an over-all restriction on its operating budget, which puts a limit on <u>total</u> expenditure for inputs and transportation. Then it is obviously the plant with the highest ratio of profit per unit of output to the sum per unit output of transportation and input costs which alone will be run.

Alternatively, let availability restrictions on some critical item (with negligible transportation cost) limit the total output of all plants. Then, of course, the plant with the highest profit per unit of product is chosen to produce the entire output.

1.5 At this point, let processing costs be introduced. These are local costs (e.g., labor costs) of processing per unit output and they may differ for different plants. Processing costs have to be subtracted in computing profits.

1.6 Next, suppose that the production capacity of each plant is limited. So long as no other restrictions enter, as in 1.1-3, this merely determines the scale of operations in plants where production is profitable. With over-all restrictions present, the set of plants operated consists of those with the highest profit-to-cost or profit-to-output ratio which still satisfy the over-all limitation.

Similar in their effect to plant capacities are availability limits on the localized resources like labor which the plant requires and whose prices determine its processing costs.

1.7 Let capacity limitations now be imposed on the resource deposits. We assume now that these are owned by the firm but that the firm pays a price for a unit of resource -- namely, its cost of extraction -- which may differ at different deposits.

Then a plant may find it profitable to obtain resources also from other deposits as soon as the capacity limit(s) of its cheapest source(s) is (are) reached. How far it should go in its search for materials is clearly indicated by the marginal cost principle: sales price in the most lucrative market must cover the cost of supply from the dearest sources that are resorted to (plus processing cost and transportation to market); the tapping of additional, increasingly dear, sources goes on until this cost can no longer be covered for any untapped source. The marginal cost of production, it will be noted, is here a step function which jumps whenever the capacity limit of some supply deposit is reached.

1.8 For an integrated firm, the supply problem is more complex, however, than the point of view of the individual plant suggests. Consumption of a resource from a deposit with short supply by a plant for which that deposit is "cheapest" may take place at an opportunity cost to other plants which use the same resource deposit. As a result it may be more profitable (to the firm) for the given plant to shift to the use of some other resource deposit if such a shift adds less to its costs per unit of its output than it subtracts from the costs of other plants who now use the resource it has liberated.

The deposits from which a given plant obtains its supplies of a resource tend to be embedded in an area --ethe supply area -- whose extension and shape is determined by the transportation cost structure and the location of rival plants. If the cost of transport per unit distance decreases with distance then it is possible that the supply area of some plant properly encloses that of another one. (In other words, the supply area may not be simply connected.)[1]

It may also be profitable to shift supplies at certain deposits from one plant to other plants without providing compensating supplies from different deposits. This amounts to a shift of production into plants of higher profitability.

If supplies of only one resource are limited, the supply from a given deposit should always go to that plant where it gives rise to the highest profit (revenue less processing costs and costs of <u>all</u> other inputs and transportation). If several resources are in short supply, the profits which are jointly attributable to the marginal shipments to a plant from deposits whose capacity limits are reached must exceed the profits forgone in the plants that otherwise would have absorbed these resources.

1.9 A more elegant description of these substitution relations is based on the introduction -- in <u>addition</u> to the "external" market prices for products and extraction costs for resources -- of prices <u>internal</u> to the firm. These are prices on (1) resources at deposits, and (2) resources at plant locations.

The internal price of a resource at a deposit equals the opportunity cost of earmarking for use in a plant a unit of the resource at the deposit. The internal price of the same resource at a plant location equals transportation cost from that deposit where this opportunity cost is smallest plus this

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[1] The theoretical analysis of market and supply areas is facilitated by a continuous model of transportation and location. Cf., Beckmann [1], [2].

smallest opportunity cost. Later the internal price of a resource at a deposit will be considered a "rent" inputed to the deposit per unit of the resource used.

Even without explicit reference to these meanings the internal prices may be defined formally as follows. The difference of internal resource prices between plant and deposit does not exceed transportation cost and equals it when shipments are actually taking place. The (external) price of product at the most lucrative market less transportation cost to the market must not exceed the cost of production in terms of internal resource prices at plant. Price and cost are equal whenever production is actually carried out. The internal price of a resource at a deposit is greater than or equal to the (external) f.o.b. price (the cost of extraction) at resource deposits and equal to it if the capacity limit of that deposit is not reached.

It takes a bit of reflection to convince oneself that these conditions are in fact all that is needed to determine the system of internal prices of outputs and flows. That these internal prices with the required properties do exist is in need of rigorous proof which will be given in the subsequent mathematical section.

1.10 The allocation of markets to plants within the firm does not raise any comparable problems. Each plant will realize its maximum profit by delivering to its most lucrative market (or if there are several, to any one of them) and this will maximize profits of the firm. Of course, one given market may be most lucrative for, and therefore served by, several plants.

1.11 Now suppose that the firm cannot determine product sales freely in each market, but is constrained to a fixed program. We may imagine that a system of maximum sales quotas and of (external) prices has been imposed by a monopolistic combination (e.g., an international cartel). Now to maximize profit, the firm still selects for each market that plant which can supply the market at the smallest opportunity cost and it fills each quota up to the point where opportunity cost becomes equal to revenue per unit, or it exhausts the quota if this cost remains below the revenue. It may be profitable now for one plant to deliver to several markets of unequal lucrativity. The most lucrative one then gives rise to an "external" profit which may be imputed as "rent" to the quota at that market and the less lucrative ones to smaller rents.

Again recourse may be had to internal prices but now, in addition, internal product prices need to be introduced. The internal product price at a plant equals the (external) product price at the market that is most lucrative after subtracting quota rent per unit product, minus the transportation cost from the plant to that market. Formally, we may again define the internal prices as accounting prices which permit at most zero ("internal") profits on production and transportation. If they are resource prices, they are equal to the (external) f.o.b. resource price at deposits, whenever the deposit capacity is not reached and are not lower otherwise. If they are product prices, they are equal to the (external) market price if the quota is not filled and are not higher otherwise. The existence of such prices and their sufficiency as criteria of profit maximization in the firm is capable of mathematical proof. (See Section 3.)

1.12 If in addition plant capacity is limited, opportunity cost is still more complex. The new limitation gives rise to one more rent, on plant or plant location per unit of output. This is to be added into the internal price of product at plant and renders it greater than or equal to its previous value, and equal if plant capacity is not reached.

The role of rent in the cost accounting of the firm is one of allocating the total profit to its sources: the resource deposits of favorable location and low f.o.b. cost, the quotas in lucrative markets, and the plants at advantageous locations. These rents have significance for long run decisions as well: a comparison of rent with the cost of capacity expansion indicates the profitability of (small scale) investments.

1.13 If a choice now exists between several production processes in a plant -- each with given local processing costs per unit output -- that process will be selected for which total cost is minimal in terms of internal prices of inputs. This amounts to a substitution of resource inputs at relatively low internal cost for more expensive ones. The situation is more complex if capacity does not assume the simple form of a ceiling on output, but of a limitation on the use of various pieces of fixed equipment. The problem becomes one of tailoring the production program to the plant facilities. As in the general theory of production we may assume that several processes may be run independently of each other in one plant. More precisely, this means that input ratios in one process are independent of the level of total output emerging from any other process. In general, it will now be a mixture of processes that is most advantageous, because application of the most lucrative process alone at its maximum level would leave certain facilities underutilized. The best combination is again characterized by zero profitability in terms of certain internal prices. With each piece of equipment which limits the capacity of plant is now associated an internal rent which is positive only when this capacity limit is effective. A set of such rents exist which together with the internal prices of other inputs will permit non-negative profits only, and permit zero profits on each process used in the optimal combination.

1.14 Finally, let us admit joint production. Again the process "levels" have to be chosen, where the level of any process is defined arbitrarily as the amount of some one output emerging. The price relations we have described remain substantially the same. The only new possibility is that it may not pay to ship a by-product to markets but is more profitable to dispose of it locally at zero price (or at a cost).

1.15 Our discussion has been concerned with the allocation of supply sources and plant capacities to outputs, given the sales price or sales program. Such allocation problems arise regardless of the organizational structure of the product market. We have shown how the level of sales in each market by a firm is determined if the price is regarded as fixed or if, in addition, sales are limited by quotas. The determination of the sales policy in less extreme cases raises difficult questions that we do not want to enter here. While allocation and sales problems cannot be disentangled completely, from a theoretical point of view, organizational considerations may in practice force a separation of the two complexes of decisions and reduce coordination between the production and sales departments to specification by the latter department to the former of fixed prices prevailing and maximum quantities saleable. In that case, the precding analysis applies whatever the sales policy is that led to the prices and quotas.

2. <u>Implications for the Industry</u>.

2.1 If the industry is made up of several firms with branch plants each owning its supply deposits and, possibly, each with market quotas, an allocation which minimizes the cost to each firm of its deliveries to markets does not insure that the cost of providing the various markets with product has been minimized for the industry as a whole. An indication of this is that the internal prices of each firm, if quoted openly as market prices for purchases and sales of (small quantities of) resources and products, might still permit arbitrage at a profit.

The allocation of resources in a firm with branch plants resembles that of an industry under competitive conditions. We permit the price determination problem, which is identical with the allocation problem, to become somewhat unusual in order to carry the analogy with the firm as far as possible. Let the demand for a product facing the industry in a given market location be perfectly elastic up to a point and thereafter be perfectly inelastic. Let all plants be rented to the firms operating them and assume that all resources are traded in competitive deposit markets. Then the previous internal prices are now market prices, profits (after paying rents) are zero for each firm; plant locations afford rents, and resource f.o.b. prices contain an element of rent reflecting the locational advantage of a deposit. The market price of product is either at or below the level at which the schedule is perfectly elastic, and below only if the saturation point of this market is reached.

The transition from the firm to the competitive industry described thus introduces no further problems. In the absence of competition principles of a different nature from those advanced here must be introduced in order to deal with the allocation of resources in the industry.

3. <u>Derivation of the Price Theorem</u>.

We now derive rigorously the internal price conditions described in Section 1 which solve the allocation problem for a profit-maximizing firm with branch plants.

We shall treat here only the case in which limitations exist on the availability of resources at various deposits and on the output capacity of plants. After the principle has been established the generalizations provide no difficulties.

3.1 Let a firm be able to sell any amount of its unique product in market locations k at price p_k. We assume that it controls a number of supply points of raw materials m, which yield maximum amounts q_i^m per unit of time, no materials being bought from or sold to the outside. The f.o.b. price of material m at deposit i is denoted by p_i^m and is not necessarily the same for all i. The firm possesses a number of plants each at a different location. The plant at location j has output capacity c_j. Costs of inputs other than materials are proportional to output; per unit of output. They are d_j, which may differ for different j. All inputs are assumed proportional to output with the input coefficients a^m the same everywhere. Let x_{ij}^m denote the flow of material m from deposit i to plant location j, and let t_{ij}, t_{jk} denote transportation costs from i to j and j to k, respectively, per weight unit; all commodities are measured in weight units.

We are interested in the quotas of sales x_{jk} allotted to plant j in market k and the levels of production $\sum_k x_{jk}$ of plant j which will maximize profit to the firm. Under the present assumptions profit equals

$$(3.1) \qquad P = \sum_{j,k} p_k x_{jk} - \sum_{i,j,m} p_i^m x_{ij}^m - \sum_j d_j x_j - \sum_{j,k} t_{jk} x_{jk} - \sum_{i,j,m} t_{ij} x_{ij}^m$$

$$= \sum_{j,k} (p_k - t_{jk} - d_j) x_{jk} - \sum_{i,j,m} (p_i^m + t_{ij}) x_{ij}^m$$

where the first term denotes net revenue from sales after processing and transportation cost, and the second term is the gross procurement cost of raw materials.

The flows are related and constrained in various ways. First, receipts of materials at plant j must equal the input requirements:

$$(3.2) \qquad \sum_i x_{ij}^m = a^m \sum_j x_{jk} \quad .$$

Next shipments from a material deposit must not exceed its yield capacity:

$$(3.3) \qquad q_i^m \geq \sum_j x_{ij}^m \quad .$$

Finally, output must not exceed the plant's capacity:

$$(3.4) \qquad \sum_j x_{jk} \leq c_j \quad .$$

In addition, we have the requirement that all the flows and production levels be non-negative:

$$(3.5) \qquad x_{ij}^m \geq 0 \qquad x_{jk} \geq 0 \quad .$$

The problem of maximizing a linear form of non-negative variables (such as (3.1)) subject to linear equations and inequalities (such as (3.2)-(3.4)) as constraints is known as an activity analysis problem. Its unknowns here include not only the production levels and sales quotas of the plants, in which we are mainly interested, but also the flows of commodities and resources between the various locations.

3.2 At this point a few general remarks on the theory of activity analysis are in order. Mathematically, linear activity analysis problems are simpler, because they are linear, but in other respects also more complex than the classical maximum problems. For the fact that inequalities occur as constraints introduces as a new, and in fact the essential, problem the determination of the set of constraints in which the equality sign is taken on.

For an activity analysis problem it is possible without actually computing the solution, to characterize it in terms of a set of equations and inequalities known as the efficiency conditions.[1] We shall discuss these for the present problem.

3.3 The efficiency conditions can be obtained formally by maximizing a new linear function of non-negative variables obtained as follows. Write all inequalities such that their left hand side is non-negative. Then add the product of these left hand terms with indeterminate non-negative multipliers to the original maximand. Add also the left hand sides of the equations after multiplication with ordinary (Lagrangean) multipliers. In the present case the required function is

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[1] These conditions are derived from the given constraints and the maximand $b'x$ in the form

(1a) $Ax \leq c$

(2a) $b' = v'A$

(3a) $v \geq 0$

(4a) $v'(Ax-c) = 0$,

where A is a matrix with as many columns as the vectors x, c, b, and 0 have components, 0 is a vector of zeros, and each of the indicated equalities and inequalities must hold for corresponding components of the vector on either side. In order that x be a problem solution it must satisfy, in addition to the constraints (1a) the inequalities (4a) for some vector v which in turn must satisfy (2a) and (3a). The efficiency conditions together with the original constraints are necessary and sufficient for x to be a solution. See Koopmans [9].

$$(3.6)\qquad \sum_{j,k}(p_r - t_{jk} - d_j)x_{jk} - \sum_{i,j,m}(p_i^m + t_{ij})x_{ij}^m + \sum_{i,m} v_i^m(q_i^m - \sum_j x_{ij}^m)$$

$$+ \sum_j v_j(c_j - \sum_k x_{jk}) + \sum_{j,m} v_j^m(\sum_i x_{ij}^m - a^m \sum_k x_{jk})$$

The multipliers v_i^m and v_j vanish whenever in the corresponding inequality the "=" sign is not assumed. This is not unexpected, for then the inequality obviously does not affect the result.

$$(3.7)\qquad v_i^m = 0 \quad \text{if} \quad \sum_j x_{ij}^m < q_i^m$$

$$(3.8)\qquad v_j = 0 \quad \text{if} \quad \sum_k x_{jk} < c_j \quad .$$

Now if a linear function $\sum_s b_s x_s$ assumes a finite maximum subject to $x_s \geq 0$ for all s as well as other constraints on the x_s, then for the maximizing x_s it must be that

$$x_s \left\{ \begin{matrix} \geq \\ = \end{matrix} \right\} 0 \quad \text{if} \quad b_s \left\{ \begin{matrix} = \\ < \end{matrix} \right\} 0 \quad \text{for all } s.$$

Applying this to the linear function (6) we have that at the required maximum

$$(3.9)\qquad x_{ij}^m \left\{ \begin{matrix} \geq \\ = \end{matrix} \right\} 0 \quad \text{if} \quad v_j^m - v_i^m - p_i^m - t_{ij} \left\{ \begin{matrix} = \\ < \end{matrix} \right\} 0$$

$$(3.10)\qquad x_{jk} \left\{ \begin{matrix} \geq \\ = \end{matrix} \right\} 0 \quad \text{if} \quad p_k - t_{jk} - d_j - \sum_m a^m v_j^m - v_j \left\{ \begin{matrix} = \\ < \end{matrix} \right\} 0 \quad .$$

In order to interpret (3.9) let us tentatively identify v_j^m with the internal price of resource m at plant j, and v_i^m with an internal rent of resource m at deposit i. The sum $p_i^m + v_i^m$ then denotes the internal price or opportunity cost of resource m at deposit location i. (3.9) therefore expresses a profitability condition for transportation of m from deposit i to plant j. In (3.10) v_j now obviously represents an internal rent on plant per unit output. For then the expression after "if" gives the profitability of production per unit output at plant j. As seen from (7) and (8) both v_i^m and v_j have a property we have required of rents: they are zero if the relevant limit is not reached.

This completes the proof (based on the indicated theorem of activity analysis) that in order for the required maximum to be attained it is necessary and sufficient that there exist internal resource prices, $p_i^m + v_i^m$ and v_j^m and internal plant rents $v_j \sum_k x_{jk}$ with the properties stated verbally in Section 1.

II. The Interregional Allocation Problem for an Economy

On the level of an economy composed of regions the short-run allocation and pricing problem must include those resources which are not specific to any industry and were treated in the analysis of the firm as available in unlimited quantities at a given price. Moreover, regional production capacity must somehow be defined and then considered fixed.

The criterion of optimization is now not as simple as for the profit-maximizing firm. Instead of focusing on a single magnitude the whole range of commodities that may be made available in all of the regions must properly be considered. Through activity analysis, however, the new kind of optimization requirement can be made to yield price conditions that clarify the allocation problem.

4. The Model.

4.1 Let the economy be divided into regions; their proper selection is a practical question with which we will not deal. Within each region we regard production as technologically uniform -- for each region there is one set of possible processes and this set may differ for different regions. As in the Leontief-Isard model (Isard [6]) it is assumed that inputs and outputs of every production process are in constant proportion for all levels at which the process is operated.

Commodities are divided into resources or primary commodities and produced goods or final commodities. Primary commodities may or may not be desirable in themselves and the same is true of final commodities; a final commodity which requires further processing is not desired in itself. Contrary to technological reality we shall treat transportation as a primary commodity desirable in itself. This will avoid some complications of the expressions to be used later without basically affecting their validity.

Not much is lost by defining a commodity unit as an "ideal weight" unit, as proposed by A. Weber [16], in order to simplify the structure of transportation costs. Otherwise it might be necessary to treat transportation costs dependent on several other dimensions as well as weight. Using "ideal weight" units, transportation cost for a unit of any commodity is given by a rate table for the originating and terminating region.

Inputs for a process may be (1) mobile primary commodities transportable between regions at a cost, possibly a prohibitive one; (2) immobile primary resources such as land and (in the short run) labor; and (3) final commodities which are transportable at a cost and are outputs of other processes. We disregard the possibility of a perfectly mobile resource, transportable at no cost, on which there is an over-all limitation for the economy. Each region may supply its requirements of a mobile resource from alternative regions as well as from itself. The "native" availability of a primary resource in a region -- the amount available per time unit without importing -- is assumed to be rigidly limited.

4.2 Our short run assumption about fixed original production capacity will be fairly realistic (given the world of perfect divisibility with which we have dealt from the start) yet will not complicate the analysis that follows. There will be several "kinds" of fixed production capacity in a region, less kinds, in general, than the number of processes available. Each fixed regional capacity is in fact an upper limit to the use of a certain kind of (perfectly divisible) fixed equipment; the limit is imposed by the regional stock of this equipment. Each capacity may be "shared" by several processes but not in general by all; the use of the same kind of equipment is an input in each such process. In the terminology just given, the use of each kind of equipment is an immobile primary resource not desirable in itself (in the short run the equipment represents a "sunk cost" and there is no "benefit" to the economy in leaving it unutilized), and the capacity which corresponds to it is its regional availability.

4.3 Now each set of process levels and interregional flows leaves each region -- after production and shipments have taken place -- with a non-negative net amount[1] of each final commodity (regional production plus imports minus exports) and a non-negative amount of each primary commodity ("native" regional availability minus regional consumption plus imports minus exports); in addition, the specified interregional flows require a total amount of transportation. The allocation problem is as follows: which sets of process levels and flows results in a "bundle" or "program" optimal in the Pareto sense and containing the total transportation cost and the net amounts left in each region of desired final commodities and of desired primary commodities? For an optimal program in this sense, given the resource availabilities, it is not possible to increase the net amount of any desired commodity left in any region without decreasing another such amount or else increasing total transportation; and total transportation cannot be decreased without decreasing the net amount of some desired commodity left in some region. Such a program we call an efficient program permitted by the resource limitations.

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[1] Wherever amounts of commodities or transportation are mentioned they are amounts per time unit. We will regard this as understood.

We may interpret the problem another way. Suppose that for the economy described it is desired to maximize national product. To compute national product the net amount of each final and primary commodity left in each region is multiplied by a selected price and the values so obtained for each region and each commodity are summed -- commodities not desired in themselves are priced at zero; from this sum is subtracted the total outlay on transportation. The relative prices and transportation charges selected reflect in some sense (with which we need not be concerned) the relative social valuation of the net availability of a unit of each desired commodity in each region and of a unit of transportation between any two regions. It is intuitively clear that when national product is maximized subject to the resource limitations an efficient program permitted by the resource limitations is achieved and conversely each such program maximizes real national product subject to the resource limitations for some set of relative prices and transportation charges.

But the goal of national product maximization for the economy is perfectly analogous to the goal of profit maximization for a single firm taking prices as given and subject only to resource limitations, a goal which we considered in the branch-plant problem. The conditions that must be satisfied if the economy (a "giant firm") is to maximize national product ("profits") -- i.e., the conditions for an efficient program -- are now familiar:

4.4 For each region there are internal prices on final commodities and mobile primary resources in addition to the "social" prices used to measure national product; there are similar internal prices in the form of imputed rents per unit on the immobile resources. These internal prices on a commodity in a region are not smaller than the social prices and are equal whenever positive net amounts of the commodity are left in the region.

Let the profits on a process in a region be calculated multiplying the amount left of each desired commodity by its internal price. Then profit must be zero on all processes actually used and no process must permit positive profit. The differences between regions of the rents on the immobile resources are subject to no further restrictions. But the internal prices of transportable commodities can differ by the cost of transportation at most. Transportation costs equal these geographical price differences wherever positive flows take place. Rents on immobile resources not desirable in themselves (whose social price is zero) exceed zero only if they are fully used up.

4.5 In order that interregional trade may take place at all, some of the internal price differences for mobile commodities must be non-zero. If the social price of each commodity is the same in all regions there may still be trade; but then differences in internal resource prices, or, for that matter, in imputed rents, can arise only out of differences in regional endowment or in the regions' distinctive technology (the processes available) which depends in turn on such things as climate, tax structure, and social institutions.

5. Price Implications of a Pareto Optimum.

5.1 The obvious question now is whether the laws governing the price structure tell the whole story of interregional allocation. The answer is in the affirmative. The price conditions are necessary and sufficient in order that an efficient program permitted by the resource limitations be attained. Moreover, using the price conditions, it is possible to compute (in principle) those flows and those regional locations and levels of processes that are required in order that either a given bundle of desired final commodities be made available in each region while forming part of an efficient program permitted by the resource limitations (if this is possible at all), or that national product be maximized subject to the limitations for preassigned social prices. We shall now formally derive the price conditions.

5.2 All commodity amounts are measured in "ideal weight" units per unit of time. Let

- i, j denote regions
- m commodities, in particular
- $\bar{m}$ commodities desirable in themselves and
- k immobile resources. Let
- n^i denote processes available in region i,
- t_{ij} transportation cost per "ideal weight" unit from i to j,
- a^{mn^i} input (if positive) or output (if negative) of commodity m per unit level of process n^i,
- c_i^m "native" availability in region i of commodity m,
- x_{ij}^m flow of commodity m from region i to region j, and
- x^{n^i} level of process n^i in region i.

Now the net availability of a transportable resource m in a region equals "native" availability (in the case of a primary resource) plus net shipments into the region (inflow minus outflow) minus requirements for production (in the case of a primary resource), or plus amount produced (in the case of a final commodity). Denoting net availability by y_i^m we have

$$y_i^m = c_i^m + \sum_j (x_{ji}^m - x_{ij}^m) - \sum_{n^i} a^{mn^i} x^{n^i}$$

For immobile resources the flow term drops out:

$$y_i^k = c_i^k - \sum_{n^i} a^{kn^i} x^{n^i}$$

Since net availabilities must be non-negative we have the constraints

$$y_i^k \geq 0 \quad , \quad y_i^m \geq 0$$

Transportation outlay equals $\sum_{ij,m} t_{ij} x_{ij}^m$. Subject to $y_i^m \geq 0$, we wish to maximize the vector $(y_i^{\bar{m}}, \ldots, -\sum_{ij,m} t_{ij} x_{ij}^m)$.

5.3 Reference must now again be made to a theorem, this time on vector maxima constrained by linear inequalities.

Theorem. Let A, B be matrices and x, c be vectors of compatible order. $Ax = \max$ subject to $Bx \leq c$ if and only if there exist vectors $u \geq 0$ and $v \geq 0$ such that

$$u'Ax + v'(c-Bx) = \max_{x \geq 0}$$

and

$$v'(c-Bx) = 0 \quad .$$

(See Koopmans [9], p. 86; cf. also p. 318).

5.4 In the present case the maximand becomes

$$\sum_{i,\bar{m}} u_i^{\bar{m}} y_i^{\bar{m}} + \sum_{i,m} v_i^m y_i^m \quad . \tag{5.1}$$

Write

$$p_i^m = \begin{Bmatrix} u_i^{\bar{m}} + v_i^{\bar{m}} \\ v_i^m \end{Bmatrix} \quad \text{if} \quad m \begin{Bmatrix} = \\ \neq \end{Bmatrix} \text{some } \bar{m}.$$

Then the efficiency conditions, obtained by differentiation of (5.1), are

$$-\sum_m a_i^{mn^i} p_i^m \begin{Bmatrix} = \\ \leq \end{Bmatrix} 0 \quad \text{if} \quad x^{n^i} \begin{Bmatrix} > \\ = \end{Bmatrix} 0 \tag{5.2}$$

$$p_j^m - p_i^m - t_{ij} \begin{Bmatrix} = \\ \leq \end{Bmatrix} 0 \quad \text{if} \quad x_{ij}^m \begin{Bmatrix} > \\ = \end{Bmatrix} 0 \tag{5.3}$$

$$p_i^{\bar{m}} > 0 \tag{5.4}$$

$$p_i^m \begin{Bmatrix} = \\ \geq \end{Bmatrix} 0 \quad \text{for} \quad m \neq \bar{m} \quad \text{if} \quad \sum_{n^i} a^{mn^i} x^{n^i} + \sum_j (x_{ij}^m - x_{ji}^m) \begin{Bmatrix} < \\ = \end{Bmatrix} c_i^m \quad . \tag{5.5}$$

The left hand side of (5.2) represents the net profitability of production activity π^i in region i. The left hand side of (5.3) similarly gives the profitability of interregional commodity shipments (arbitrage). The prices $\bar{p}_i^m$ of all commodities desirable in themselves are positive (5.3) (because of the $\bar{u}_i^m$- term they contain). Other resources are free goods, if not used up to capacity; otherwise they afford a scarcity rent. In terms of our national product interpretation the u's are social prices and the p's internal prices.

6. The Invariance of Input Coefficients.

We now digress briefly from our short run assumptions in order to present a further result of some interest.

6.1 Suppose that there is only one primary resource labor, which must now be regarded as perfectly mobile among regions. In particular, the commodity transportation must be further broken down into labor inputs and inputs of intermediate commodities: a fixed amount of each is required to transport a commodity unit from one region to another. Let there be no joint production. Then the application of a well-known theorem in activity analysis (Samuelson [14]) has the immediate consequence that all efficient programs permitted by the available labor supply can be achieved by using the same processes and the same supply regions (for any commodity unit in any region there is a supply region from which the commodity unit came); and these same processes and supply regions achieve all such efficient programs as the supply of labor is changed. In particular, for each final product just one invariant process may be chosen and each commodity may be supplied to a given region by just one invariant region.

Under these conditions the input coefficients as observed at any one time are constants (as long as no technological discoveries are made). It is justified therefore to base predictions of future process levels and interregional flows on these coefficients.

6.2 To some extent the theorem can be reversed. If there are more than one primary resource and any one of these gets into short supply in a region a change of the processes or supply patterns becomes necessary as the program is changed. In this case, there is no theoretical justification for regarding the prevailing input coefficients as constant, although practical arguments may be advanced for doing so (Isard [3], [4]).

7. A Balance of Trade Equation.

7.1 An important dividing line still exists between the interregional allocation model and a model describing the short run equilibrium of an interregional economy. We have taken no account of the requirement that all commodity quantities received must be purchased out of the incomes that accrue to the factors in each region.

Every household is subject to a budget constraint. After aggregation over the region these constraints imply a balance of trade equation. Considering the simplest case, let all households or factors earn as well as consume their income in one region, the region of "native" residence, (they may export their services to another region but the income earned is regarded as earned "by" their "native" region). Factors -- or rather factor services -- are regarded as primary commodities not desired in themselves; in fact, let us call all such commodities factor services. Under stationary conditions total income is consumed. We first note incidentally that the aggregate income of factors in the region i, call it y_i, cannot exceed the income when all factors are fully employed:

$$(7.1) \qquad y_i \le \sum_{m \neq m} p_i^m c_i^m \quad .$$

Now the total value of commodities that becomes available for consumption in the region (factors cannot consume factors) is

(7.2) $$e_i = \sum_{\bar{m}} p_i^{\bar{m}}[-\sum_{n^i} a^{\bar{m}n^i} x^{n^i} + \sum_j (x_{ji}^{\bar{m}} - x_{ij}^{\bar{m}})].$$

Regional income equals the difference between the value of factor exports and the value of factor imports plus the value of factor inputs in the region:

(7.3) $$y_i = \sum_{m \neq \bar{m}} p_i^m[\sum_{n^i} a^{mn^i} x^{n^i} + \sum_j (x_{ij}^m - x_{ji}^m)] \quad .$$

Equality of regional income and consumption now implies

(7.4) $$y_i = e_i \quad .$$

Then, because of (7.2) and (7.3)

(7.5) $$\sum_m \sum_{n^i} p_i^m a^{mn^i} x^{n^i} + \sum_m \sum_j p_i^m (x_{ij}^m - x_{ji}^m) = 0 \quad .$$

Now the first term is zero by (5.2). Thus we obtain the regional balance equation

(7.6) $$\sum_m p_i^m \sum_j (x_{ij}^m - x_{ji}^m) = 0 \quad .$$

This is a further condition to be satisfied by a set of prices and flows satisfying the efficiency conditions, if these prices are actually to prevail in the economy and if equilibrium as well as an efficient program permitted by the resource (factor) limitations is to obtain.

7.2 We shall not write out the more complicated equations that result if each region is spending a fixed proportion of its income in other regions and secures income from factor ownership in other regions. Suffice it to say, that the unbalance between the values of physical inflows into and outflows out of a region must be offset by flows of money across regional boundaries to owners of resources and to the locations they choose for consumption.

Conclusion

We have seen that in a short run approach some of the peculiar difficulties that beset location theory may be skirted and recourse may be had to the price theoretical methods of allocation and equilibrium analysis which activity analysis provides. This sheds light on the geographical distribution of observed prices and clarifies the role of internal or accounting prices as guides to allocation. Substitution takes the form of a variation of processes and of supply and market points. Because of short run rigidities in plant and market capacities there may not be freedom of substitution in all directions: the profit maxima occur at the edges of the possibility set, mathematically speaking, and this necessitates efficiency conditions in terms of marginal inequalities as well as equations.

But the resulting conditions of allocative optimization and equilibrium are intuitive: possible profits must not be positive and must equal zero in efficient activities. These are conditions that hold "in the small" and may be verified independent of each other. At the same time they are sufficient conditions, and this, it may be noted, is due to the fact that, in spite of discontinuities in the substitution relations, the law of decreasing returns to substitution is maintained. Details may be found in the theory of activity analysis on which this approach has been based.

References

[1] Beckmann, M., " A Continuous Model of Transportation", _Econometrica_, (1952), pp. 643-660.

[2] ____________, "The Partial Equilibrium of a Continuous Space Market", _Weltwirtschaftliches Archiv_, 71 (1953), pp. 73-90.

[3] Furlan, V., "Die Standortsprobleme in der Volks- und Weltwirtschaftslehre", _Weltwirtschaftliches Archiv_, 2 (1913).

[4] Hitchcock, F. L., "The Distribution of a Product from Several Sources to Numerous Localities", _Journal of Mathematics and Physics_, Massachusetts Institute of Technology, 20 (1941), pp. 224-230.

[5] Isard, W., "Distance Inputs and the Space Economy", _Quarterly Journal of Economics_, 65 (May, 1951) pp. 181-198 and (August, 1951) pp. 373-399.

[6] _________, "Output Analysis: A Model of a Space Economy", _Review of Economics and Statistics_, 33 (1951), pp. 318-328.

[7] _________, "A General Location Principle of an Optimum Space-Economy", _Econometrica_, 20 (1952), pp. 406-430.

[8] _________, "Regional Commodity Balance and Interregional Commodity Flows", _American Economic Review_, 43 (1953), pp. 167-180.

[9] Koopmans, T. C., "Analysis of Production as an Efficient Combination of Activities", in _Activity Analysis of Production and Allocation_, New York: John Wiley and Sons, 1951, pp. 33-98.

[10] ______________, and Reiter, S., "A Model of Transportation", _ibid_., pp. 222-260.

[11] ______________, "Optimum Utilization of the Transportation System", _Econometrica_, 17, Suppl. (July, 1949), pp. 136-46.

[12] Loesch, A., _Die räumliche Ordnung der Wirtschaft_, Jena: Gustav Fischer Verlag, 1st ed.; 1940; 2nd ed., 1944.

[13] Predoehl, A., "Das Standortsproblem in der Wirtschaftstheorie", _Weltwirtschaftliches Archiv_, 21 (1925), pp. 294-321.

[14] Samuelson, P., "Abstract of a Theorem Concerning Substitutability in Open Leontief Models", in _Activity Analysis of Production and Allocation_, New York: John Wiley and Sons, 1951, pp. 142-146.

[15] Samuelson, P., "Spatial Price Equilibrium and Linear Programming", American Economic Review, 42 (1952), pp. 283-303.

[16] Weber, A., Ueber den Standort der Industrien, Tubingen, 1904. Translated as Alfred Weber's Theory of the Location of Industries, Chicago: University of Chicago Press, 1928.