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Communication in Networks[1/]

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I. Introduction

In some approaches to the study of organization the problem of determining the costs of achieving the exchange of information between organization members is given a central role (see, for example, [1]). Certain aspects of this communication problem lend themselves to precise formulation in terms of the algebra of matrices of zeros and ones, or alternatively, in terms of graph theory. Our purpose here is to discuss such a formulation and briefly review that literature on the subject which has come to our attention.

II. Problem Formulation

Consider a group of people $1,\ldots\ldots,r$, each of whom may observe certain pieces of information x_j $(j=1,\ldots,t)$, and communicate information to certain others in the group. Let Ω be a matrix of elements w_{ij} such that w_{ij} equals 1 or 0 according as x_i is or is not observed by person j (Ω need not be square). Furthermore, let R be a square matrix of elements r_{ij} such that $r_{ii} = 1$ and r_{ij} equals 1 or 0 according as i can or cannot communicate to j. Think of the communication taking place in "pulses" so that at each

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pulse each person i communicates all the information which has thus far reached him to all persons j for which $r_{ij} = 1$. We can describe the information each person has after n pulses by a matrix $S(n)$ with elements $s_{ij}(n)$ such that $s_{ij}(n)$ equals 1 or 0 according as person j has or has not received the information x_i on or before the n'th pulse. If we adopt the (Boolean) arithmetic $1+1 = 1 + 0 = 1 \cdot 1 = 1, 0 + 0 = 0 \cdot 0 = 0 \cdot 1 = 0$, and retain the usual rules for matrix multiplication, then it is easily verified that

$$S(n) = \Omega R^n$$

(Unless otherwise noted all matrix multiplication in this paper will be of this Boolean type).

Any t by r matrix of ones and zeros may be thought of as describing what information the various persons have about the x_i; that is, we can say that person j knows or does not know x_i according as s_{ij} equals 1 or 0. Any such pattern of information (or the corresponding matrix) will be called an <u>information structure</u>.

Our general aim is to determine how to achieve any given information structure in the cheapest possible way. A structure S can be achieved by an <u>observation matrix Ω</u> and a <u>network matrix R</u> in n pulses if

$$\Omega R^n \leq S$$

(we say $A \leq B$ if $a_{ij} \leq b_{ij}$ for all i,j). The costs of doing this may be associated with Ω, with the network R, and with the number of pulses n. We may denote the total cost by $C(\Omega, R, n)$. Thus our general problem is:

> Given an information structure S and a cost function $C(\Omega, R, n)$, find Ω, R and n so that $C(\Omega, R, n)$ is minimized subject to $\Omega R^n \leq S$.

Without further specification of the cost function it seems doubtful that any progress can be made in the above problem. One simple function that suggests itself is linear in $\sum_{ij} w_{ij}$ (the number of observations), in $\sum_{i \neq j} r_{ij}$ (the number of one way links in the network) and in n.

If we concentrate on the time factor a special problem can be formulated whose solution might give some insight into the general problem. Let $n_o(\Omega, R)$ be the smallest integer n such that

$$\Omega R^n = \Omega R^{n+1}$$

ΩR^{n_o} is thus the "maximal" information structure that can be achieved with Ω and R, and n_o is the smallest number of pulses in which this can be done*. Our special problem is:

Given Ω and R is there a simple way to find n_o and ΩR^{n_o}?

In all of these problems the case in which Ω is the identity matrix I is of special interest, and leads to substantial simplifications. Any situation in which the pieces of information initially available to the different persons are determined and distinct comes under this case.

III. The Literature

1. An Alternative Formulation

The problem is given a slightly different formulation by D. Rosenblatt [2]. He uses the same matrix representation of the communication process, but thinks in terms of three processes; that is, there is a matrix A representing the way ideas (or attitudes, rumors, or orders) affect each other without the presence of a formal organization to communicate them, there is a matrix B to represent communication among members of the organization, and there is a matrix C to represent the process by which the people acquire the ideas.

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* That n_o always exists follows from the easily seen fact that for every n, $\Omega R^n \leq \Omega R^{n+1}$. This latter depends upon the assumption that $r_{ii} = 1$ for all i. (i.e. R is reflexive).

In one time-interval the information structure is C (as people have just had time to pick up ideas), in two time-intervals it is AC + CB (since ideas could affect each other for one interval and then go to people, or go to people and be transmitted once by them), in three time-intervals it is $A^2C + ACB + CB^2$, etc. If U is the information structure at one stage, then AU + UB is the structure at the next. Assuming again that A and B are reflexive we see that here also a maximal structure is arrived at in a finite number of steps. Mr. Rosenblatt gives an algorithm for computing the minimum time it takes for any given individual to acquire any given bit of information, provided you know each information structure up to and including the final one. It appears that this algorithm is too tedious to be of practical value.

2. The Solution of Boolean Matrix Equations

R. Duncan Luce [3] gives a necessary and sufficient condition, easily applied except in very large organizations, for the existence of a solution to the Boolean matrix equation XA = B. This is applicable if one would like to find out whether there exists an n, given R and S, such that $R^n = S$. Certainly XR = S has a solution if such an n exists, so this gives us a necessary condition that a given network induce a given structure or a given structure be induced by a given network. The condition is simply that XR = S has a solution if and only if $S \leq (S'R^T)'R$, where M' is the matrix that has 1 where M has 0 and has 0 where M has 1, and M^T is the transpose of M. If X exists, then $X = (S'R^T)'$ is a solution.

3. Graph-Theoretical Approach

A number of authors, Luce included, view the problem in large part as one in graph theory, and they try to use graph-theoretical methods that have some intuitive sociological meaning. Luce and Perry [4] refer to "cliques" and "n-cliques"; a clique is a maximal collection of individuals in an organization who can each send messages directly to each of the others, an n-clique generalizes a clique to the extent that in an n-clique the communicating between members need not be immediate but may (and for at least one pair of members, must,) take n steps; a clique is a 1-clique. R. S. Weiss [5] considers the degree of "cohesiveness" (in an intuitive sense) of an organization and considers intuitively appealing ways in which to analyze an organization into subgroups and liaison individuals, a subgroup being a collection of individuals where "almost all" possible links between individuals in the subgroup exist and "almost none" to other sub-groups, except for liaison individuals, who make the whole more cohesive by supplying inter-group contacts. The concepts of "sub-group" and liaison individual are not strictly defined, and the "algorithm" Weiss gives for such an analysis therefore does not lead to a unique division of the group. R. P. Abelson [6] introduces the concept of a net, which is a maximal collection of members of an organization who can each communicate eventually with each of the others. Between two distinct nets messages can travel in only one direction though perhaps over more than one path. (See section IV for some further remarks on nets.)

4. Costs and Inefficiency

A. Shimbel [7] suggests three possible measures of cost or inefficiency in an $r \times r$ communication network R. Let n_o be the first integer n cuch that $R^n = R^{n+1}$, let $S = R^{n_o}$ in the Boolean sense (maximal information structure), and let $S^* = R^{n_o}$ in the ordinary (non-Boolean) sense. The three measures of inefficiency are:

(i) n_o

(ii) $\frac{1}{r} \sum_{i \neq j} r_{ij}$ (average number of one way links per person)

(iii) $\sum_{i,j} s^*_{ij} \Big/ \sum_{i,j} s_{ij}$

The first two of these have already been mentioned in section II (the case in which $Q = I$). The third is a measure of the redundancy in the system. (See sec. IV for some further remarks on redundancy).

IV. Additional Remarks

1. Nets

The concept of a net introduced by Abelson (See sec. III. 3) is useful in analysing the maximal information structure matrix $S = R^{n_0}$ of a network matrix R. Let us renumber the persons so that persons in any one net have consecutive numbers, and let us label the nets N_1, N_2, etc. We note that if a member of N_i can (directly or indirectly) send messages to a member of N_j then every member of N_i can send messages to every member of N_j; hence the blocks obtained by partitioning S according to nets each consist entirely of 1's or entirely of 0's. If N_i can send to N_j and N_j can send to N_i we must have $i = j$. Thus there exists a reordering of the nets such that all blocks of S below the diagonal consist of 0's. We see lastly that if the block in the i-th row and j-th column and the block in the j-th row and k-th column are both composed of 1's then so also is the block in the i-th row and k-th column, that is, the matrix is transitive as regards blocks.

It is easily seen that a matrix S with 1's on the diagonal can be a structure matrix if and only if S is transitive, or alternatively, if and only if $S = S^2$.

2. Redundancy

As Shimbel points out, the element s^*_{ij} of S^* (see sec. III. 4) does not equal the number of times the information x_i has been communicated to person j. This latter number is actually given by the ij element of the matrix T which is defined as follows:* Let $S(n) = R^n$ in the Boolean sense

* The observation of x_i by person i will also be counted as a communication.

and let $T(n) = I + \sum_{k=0}^{n-1} S(n)\ [R-I]$ in the ordinary sense. $S(n)$ is the information structure after n pulses and hence the ij element of $S(k)\ [R-I]$ (ordinary sense) gives the number of times person j receives information x_i on the $(k+1)$st pulse. Thus the ij element of $T(n)$ gives the total number of times x_i has been communicated to person j in n pulses. Define $T = T(n_o)$.

Thus a more appropriate measure of redundancy is

$$\sum_{i,j} t_{ij} \Big/ \sum_{i,j} s_{ij}$$

Another quantity of interest is the number of distinct paths by which x_i gets to person j*. If we let $R = I + Q$, then it is well known that this number is given by the ij element of

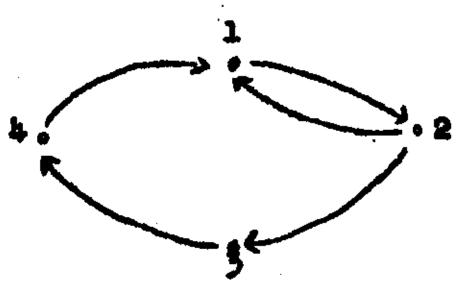
$$U = I + Q + Q^2 + \ldots + Q^{n_o} \quad \text{(ordinary sense)}$$

In fact the ij element of Q^n is equal to the number of distinct paths by which x_i gets to person j in exactly n steps.

For example, let R be the 4×4 matrix

$$R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

This corresponds to the graph:



* The "null path" by which x_i gets to person i in zero steps will also be counted here as one path.

In this case $n_o = 3$ and the final information structure matrix has all its elements equal to 1. Also:

$$S^* = \begin{bmatrix} 4 & 4 & 3 & 1 \\ 5 & 4 & 4 & 3 \\ 3 & 1 & 1 & 3 \\ 4 & 3 & 1 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 3 & 3 & 2 & 1 \\ 4 & 3 & 3 & 2 \\ 2 & 1 & 1 & 3 \\ 4 & 2 & 1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 3 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}$$

Shimbel's measure of redundancy is $\frac{45}{16} = 2.81$ whereas the corresponding measure proposed here is $\frac{36}{16} = 2.25$.

References

[1] J. Marschak and R. Radner, "The Firm as a Team", CCDP: Economics No. 2093 (dittoed).

[2] D. Rosenblatt, "A Note on Communication in Organizations", Carnegie Institute of Technology, 24pp. (dittoed).

[3] R. D. Luce, "A Note on Boolean Matrix Theory", Proc. Am. Math. Soc., 3 (1951), 382-388.

[4] R. D. Luce and A. D. Perry, "A Method of Matrix Analysis of Group Structure", Psychometrika, 15 (1950, 169-190.

[5] R. S. Weiss, "A Sociometric Analysis of the Structure of Role Relations in a Government Organization", Univ. of Michigan Seminar on the Application of Mathematics to the Social Sciences (dittoed).

[6] R. P. Abelson, "Mathematical Contributions to Group Psychology", Master's Thesis, M.I.T., Dept. of Math., Feb. 1950.

[7] Shimbel, A., "Applications of Matrix Algebra to Communication Nets", Bull. Math. Biophys., 13 (1951), 165.