COWLES COMMISSION DISCUSSION PAPER: ECONOMICS NO. 2097



# Mathematical Models and Experiments on Decision Making.

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February 15, 1954

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*) Preliminary outline of a paper to be presented in New York, February 26-27, 1954 (excepting possibly Sections VIII and IX), at a meeting, sponsored by the Armed Forces Epidemiological Board, Office of the Surgeon-General, U. S. Army and Dunlap and Associates, Inc., and devoted to Mathematical Models and Human Behavior. With acknowledgments to Roy Radner, Cowles Commission; to Ward Edwards, Johns Hopkins University; Conrad Oort, Fellow of Committee on Behavioral Sciences, University of Chicago; T. W. Anderson, Columbia University.

Research under Contract with the Office of Naval Research, Nonr-358(01) (NR-047-006).

## I. NORMS, HABITS, and TEACHING (and SELECTION) METHODS.

1.1. We can prescribe norms; describe habits; and teach habits that approach norms. Analogy:

| | | |
|---|---|---|
| Norms: | Rules of arithmetic, geometry, logic | Formal norms of good decision making. |
| Habits: | Actual problem solving by children, normal adults, abnormal adults, members of primitive tribes, etc. | Actual decision making. |
| Teaching methods: | Methods of Teaching mathematics, etc. | Training or selecting for formally good decision making. |

1.2. There is a social need for both observing and improving decision habits. Hence the interest of the social scientist in all three: the norms, the habits and the methods of training and selecting.

## II. SURE vs. UNCERTAIN PROSPECTS.

2.1. The outcome function (matrix): outcome depends on situation (column) and decision (row): $a = a(d_i, s_j)$.

2.1.1. More generally, outcome depends on the time-sequence of situations and on a sequential strategy. The outcome itself may be a time-sequence.

2.2. Prospect: a row of the outcome matrix: $a_i = [a(d_i, s_1), a(d_i, s_2), \ldots]$.

The case of sure prospects: outcome depends on decision only, since the situation is unique and known. Hence $a_i = a(d_i)$; a decision $d_i$ can be represented by the (sure) prospect $a_i$ it entails.

## III. SURE PROSPECTS: Norms.

3.1. Ordering of achievable sure prospects.

(a) transitivity of the ordering relation

(b) universality of the ordering relation

(c) constancy of the ordering relation.

(a) and (b) and (c) = constant complete ordering (ranking).

3.2. Ordering of "differences" between prospects.

(d) constant complete ordering of differences.

(a) and (b) and (c) and (d) = constant scaling ("unique up to linear transformation").

IV. SURE PROSPECTS: Habits.

4.1. Stochastically constant rankings.

4.1.1. Model A. Let $\{a_1, \ldots, a_n\}$ be the set of achievable sure prospects and let $a_h R a_i$ mean that, in a particular trial, the subject has ranked $a_h$ above or on par with $a_i$. Then there exists a n-variate distribution $F(U_1, \ldots, U_n)$ such that the probability of the ranking $a_h R a_i R a_j \ldots$ equals the probability (computable from F) that $U_h \geq U_i \geq U_j \ldots$ .

4.1.2. Predictive use of Model A.

4.1.3. The case of unlimited samples (many trials with many subjects having identical distributions F): verification of Model A and measurement of F.

4.1.4. The case of limited samples. Testing of A and estimation of F.

4.1.5. Model B. Restrict the distribution F of Model A as follows: 1) there exists a unique most probable ranking of the n prospects; 2) the probability of a given ranking is the smaller the larger the number of "reversals" sufficient to obtain this ranking from the most probable ranking (i.e., the smaller the Kendall coefficient of agreement).

4.1.6. What is the use of such restrictions for estimation and prediction?

4.2. Applying the Thurstone-Mosteller scaling method to prospects.

4.2.1. Model C. Restrict the distribution F of model A as follows: F is normal with all variances ($\sigma^2$) equal and all correlations ($\rho$) equal. The means $E\ U_i = u_i$ form a scale.

4.2.2. Moreover, the model can be reformulated as follows: There exists a normal univariate distribution G such that $G(u_i - u_j)$ is equal to the probability that $a_i R a_j$.

4.2.3. Questions: What is the predictive use of scaling of prospects: What restrictions on F are sufficient and necessary to generate a scale?

4.2.4. An important restriction on models A, B, C in the case of quantifiable "goods" and "bads." Suppose $a_1$ means one meal, and $a_2$ means two meals. Then (unless the cost of throwing a meal away is considerable) we shall expect $a_2 R a_1$ with certainty for all people, with only pathological exceptions. This can also be extended to some vectors of quantities of goods; e.g., if $a_1$ = one meal and two theatre tickets and $a_2$ = two meals and three tickets then $a_2 R a_1$ with certainty.

This restriction rules out the normality assumption of Model C, since that assumption does not admit, for any pair $a_h$, $a_i$ that the (marginal) probability of $a_h R a_i$ be 1 (or 0). Even a weaker model than Model C is inconsistent with choices among varying quantities of a "good." Any model that assigns an equal scale difference $u_h - u_i$ to all pairs $(a_h, a_i)$, $(a_j, a_k)$,... for which the probabilies of $a_h R a_i$, $a_j R a_k$ etc. are equal, will lead to an absurd result whenever $a_h$, $a_i$, $a_j$, $a_k$, ... are decreasing quantities of a good. Then the three probabilities of $a_h R a_i$, $a_i R a_j$ and $a_h R a_j$, respectively, are all equal (viz., =1) and hence $u_h - u_i = u_i - u_j = u_h - u_j = 0$.

We shall encounter this difficulty again in the case of uncertain prospects.

Note also that if the quantities of some goods are continuous, the distribution F becomes infinite-dimensional.

V. SURE PROSPECTS: Teaching.

5.1. A goal. Our economy needs men able "to adjust easily to change, to make sound decisions quickly and firmly" (TIME).

5.2. Two decision-process models:

a) Static. The distribution F of model C is assumed constant. A "quick and firm" decision maker is characterized by a small $\sigma^2$, for a given set of alternative prospects. $\sigma^2$ is a measure of "hesitation."

b) Dynamic. Let t be the time elapsed (or the number of trials made) after the subject was present with a set of alternatives. Then F depends on t; and in particular $\sigma^2$ decreases with t. Specification: $\sigma^2(t) = \sigma^2_\infty + \sigma^2(0) \cdot e^{-\alpha t}$, $\alpha > 0$. A "quick and firm" man has small $\sigma^2_\infty$ and large $\alpha$, or small $\sigma^2(0)$.

5.3. Problem of Training (or vocational guidance): find methods of filling decision-making positions with people who (among other characteristics with which we are not concerned here) are "quick and firm."

VI. UNCERTAIN PROSPECTS, WITH NO PROBABILITIES INVOLVED: Norms.

6.1. If the outcomes are completely ordered, the decisions are at least partially ordered. If $a(d_h, s_j)Ra(d_i, s_j)$ for all j, then $d_h$ and $d_i$ are comparable: $d_hRd_i$. If $d_hRd_i$ and not $d_iRd_h$, $d_i$ is dominated by $d_h$.

6.2. For every subject, there exists a complete ordering of decisions. This makes the theory of decision making under uncertainty quite analogous to the pure economics of consumption: in neither case is an attempt made to explain a particular ordering, but certain verifiable theorems are derived. Debreu has shown that the economic theorems of equilibrium and welfare optimum under perfect market conditions extend to the case of uncertainty thus formulated.

VII. UNCERTAIN PROSPECTS, WITH NO PROBABILITIES INVOLVED: Habits.

7.1. A stochastic model for 6.1: the probability for not rejecting a dominated decision is non-zero. This is hardly useful outside of kindergartens (and mental asylums) -- but it may indeed be useful in kindergartens, to measure progress in the first steps of training for good decision making.

7.2. Stochastic models for 6.2: these would be analogous to A, B, C in Section IV. Note that if the number of states is infinite (e.g., if the states are characterized by continuous parameters), F becomes infinite-dimensional.

7.2.1. A restriction analogous to 4.2.4 obtains when some decisions are dominated. The probability of their rejection is 1, at least after some thinking by a normal civilized adult, or after some training of a child or primitive.

VIII. UNCERTAIN PROSPECTS, WITH PROBABILITIES INVOLVED: Norms, Habits.

8.1. Defining subjectively equal chances of two events.

Let the subject consider (or imagine) ten dogs aged respectively 1, 2, 3,...,10 years (and numbered 1, 2, ... 10). Let him consider, in some random succession, ten pairs of offers:

(1) To get $100 if the i-th dog lives a year from now, nothing if it dies.

(2) To get $100 if the i-th dog dies within a year, nothing if it lives.

Suppose the subject prefers to bet on the life of the dogs aged less than 6;

against the life of the dogs aged more than 6; and is indifferent in the case of the six year old dog. Then his personal probability f that dog's survival is defined as $\frac{1}{2}$. For the two groups of other dogs, his personal probabilities are $<\frac{1}{2}$, and $>\frac{1}{2}$, and not yet further specified.

This definition is, in essence, due to Ramsey (as is the Norm that follows).

8.2. Consistency of subjectively equal chances: Norm.

This Norm asserts: the subject's view of the probabilities should not depend on the rewards consequent upon his action. Thus, if in the offers listed above, the rewards "$100 or nothing" would be replaced by another pair of distinct rewards -- e.g., "$10 or a trip to Florida" -- he should still be indifferent in the case of a 6 year old dog (and, in general,*) be not indifferent in the case of all other dogs): the chances of that dog's survival or death must still be equal.

8.3. Habits and subjective probabilities.

We can expect that, as in the case of all other rankings discussed previously, the preference rankings establishing subjective probabilities will exhibit "hesitations," and an appropriate stochastic model can be tested and estimated. Of particular interest would be a dynamic model, possibly indicating how a person converges to some view on subjective probabilities.

A different matter is the testing of the consistency norm -- i.e., observing whether the preferences are affected by a change in rewards. Perhaps habits can be brought in line with the consistency norm by easy training -- essentially by training the subject to concentrate.

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*) In a special case, the two rewards, though physically distinct, may make the subject indifferent between the two offers, for any age of the dog. In this case, the utilities of the two offers (e.g. a trip to Florida and a trip to California) will be equal, as will become clear below.

It seems to me that much of the behavior of an inexperienced gambler is similar to the behavior of an inept solver of a problem in school geometry. I watched a lady in Reno, Nevada, whose roulette game consisted in always putting single silver dollars on each of 18 squares (out of the 36 available squares) of the table instead of putting $18 on a single place called "red" or "odd". This would produce exactly the same result with fewer manipulations. Perhaps these manipulations (and the sight of a silver pile won on one square, though offset by losses on 17 others) gave her pleasure. More likely, she had not stopped to think.*)

8.4. Extension to probabilities other than $\frac{1}{2}$.

The outcome matrices used to define equal chances were as follows (with a and b denoting a pair of rewards, such as "$100" and "nothing"):

| | Survival of i-th dog | Death of i-th dog |
|---|---|---|
| Bet on survival | a | b |
| Bet on death | b | a |

The subjective probability of the survival of the i-th dog was defined to be $>\frac{1}{2}$, $=\frac{1}{2}$, or $<\frac{1}{2}$ according to whether the subject prefers to bet on its survival, is indifferent, or prefers to bet on its death. The consistency norm says that

*) Apparently some subjects observed systematically by Mr. Nogee showed the same bias; they justified it by some vague verbal explanations. (Mimeographed report of a paper read at the University of Michigan on 19 November 1953.)— Fedor Dostoyevsky, one of the most famous and most passionate gamblers was a victim of several fallacies (as shown in his letters and his novel, The Gambler). For example, he believed that after a long sequence of "blacks" the chance of a "red" was larger than ordinarily. He also seems to have believed in what is today called psycho-kinesis; but this, if a fallacy, would not be one that transgresses logical norms. —

the same rankings should prevail when the pair of rewards a, b were replaced by two other distinct rewards.*)

The following extension is possible (in the spirit of L. J. Savage):

| Decision \ Events | $s_{i1}$ | $s_{i2}$ | $s_{i3}$ |
|---|---|---|---|
| $d_1$ | a | b | b |
| $d_2$ | b | a | b |
| $d_3$ | b | b | a |

If i is chosen so that the subject is indifferent between the three events $s_{i1}$, $s_{i2}$, $s_{i3}$, then their subjective probabilities are said to be $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$. In a similar way, the definition, and the corresponding consistency norm is applied to all rational fractions. We mention this approach as a challenge to designers of experiments, more complex than those needed to establish subjective probabilities equal to $\frac{1}{2}$.

8.5. Equidistant utilities: definition, norm, habits.

Suppose we know that the subject has assigned probabilities $\frac{1}{2}$ to two alternative events, $s_1$ and $s_2$. Let the matrix of outcomes contain the following rows:

| | | $s_1$ | $s_2$ |
|---|---|---|---|
| decision | $d_1$ | a | b |
| decision | $d_2$ | c | d |

*) Except for a special case in which the two bets become indifferent for every i.

Suppose that we succeed to adjust the rewards a, b, c, d in such a way that the subject is indifferent between $d_1$ and $d_2$. We then say that for him the following utility differences are equal:

$$u(a) - u(c) = u(d) - u(b):$$

[in words: if a is better than c, then b must be "correspondingly" worse than d, in order to make a 50-50 chance of a and b no better and no worse than a 50-50 chance of c and d].

This is a definition. Ramsey's corresponding (verifiable) consistency norm says that the said utility differences should remain equal if $s_1$ and $s_2$ were replaced by some other pair of alternative events, $t_1$ and $t_2$, for which the subject had assigned probabilities $\frac{1}{2}$, $\frac{1}{2}$ (in the sense of our previous sections). That is, if in the above matrix, $s_1$ and $s_2$ are replaced by $t_1$ and $t_2$ (or $t_2$ and $t_1$) the subject should remain indifferent between the decisions $d_1$ and $d_2$.

In testing whether such consistency is habitual and/or teachable, one may, for simplicity, make b and d identical. Also, the consistency of "equal chances of events" (independent of outcomes) and the consistency of "equal differences of utilities" (independent of events) might be tested simultaneously, from one batch of experiments. Lives of dogs, cats, humans; elections; surgical operations; weather, — provide adjustable alternatives of events. Sums of money and various single sources of pleasure and pain (possibly combined into vectors), provide adjustable rewards.

8.6. <u>Utility scale and expected utility: Norms.</u>

8.6.1. <u>Ramsey's procedure.</u> If the consistency norms just treated are satisfied, the differences between utilities of my two outcomes are completely ordered — as in Section 3.2 above. Utilities can then be scaled. The next step on Ramsey's route is to establish probabilities other than $\frac{1}{2}$. Let the outcome matrix be

| | $s_1$ | $s_2$ |
|---|---|---|
| $d_1$ | a | a |
| $d_2$ | b | c , |

the utilities u(a), u(b), u(c) having been scaled as previously indicated, with $u(b) \neq u(c)$. If the subject is indifferent between $d_1$ and $d_2$, the probabilities $p(s_1)=p$ and $p(s_2)=1-p$ are <u>defined</u> by

$$(8.A) \qquad u(a) = u(b) \cdot p + u(c)\ (1-p),$$

$$p = \frac{u(a)-u(c)}{u(b)-u(c)} \quad .$$

That is, Ramsey's subjective probability of an event ($s_1$) is defined by the odds at which the subject would bet on that event. This definition is as old as Bayes, who, however, used money gains rather than utilities.

The corresponding <u>consistency norm</u> says that p, 1-p should remain the same, if another triplet, (a', b', c'), were substituted for (a, b, c), such that the subject be indifferent between $d_1$ and $d_2$. — Again, the testing of the habits with regard to this third consistency norm could be made on the basis of the same batch of trials.

8.6.2. <u>Ramsey and Bernoulli</u>. Clearly, this last mentioned consistency norm is identical with the Daniel Bernoulli norm "Maximize expected utility." The right hand member of equation (8.A) is the expected utility of the prospect "b if $s_1$, c if $s_2$." If another prospect a' would have expected utility $u(a') > u(a)$, it would be preferred. The extension to the case of more than two events is obvious.

8.6.3. Another way is that of Savage (and De Finetti): as indicated in 8.4, one might establish probabilities other than $\frac{1}{2}$ without first scaling utilities. The Bernoulli norm can then be derived (as will be shown in Section IX), and the utilities scaled accordingly.

8.6.4. Rules of compounding subjective probabilities (of events $s_1$ <u>and</u> $s_2$; of $s_1$ <u>or</u> $s_2$; of $s_1$ <u>given</u> $s_2$) are the same as the classical ones. This was known to Bayes and has been again shown by Ramsey as well as Savage.

8.6.5. <u>Role of "objective probabilities."</u>

"Objective probabilities" matter only to the extent to which they influence subjective ones. In the case of familiar or even less familiar random devices with simple mechanical properties (dice, coins, decks of cards), the subject will soon learn to make his subjective probabilities approach the "objective" ones -- and the psychological processes deserve studying (they have been studied by Mosteller and Nogee).

## IX. <u>A SEMI-VERBAL "PROOF" OF BERNOULLI NORM.</u>

9.1. I want to convince the reader that it will be possible for him to scale his utilities of all prospects, and that the scale will satisfy the following property: if a prospect x is a lottery promising the prospects $a_1, a_2, \ldots, a_n$ with subjective probabilities $\pi_1, \ldots, \pi_n$, then

$$(9.A) \qquad u(x) = \sum_1^n u(a_i)\,\pi_i.$$

9.2. Let a and b be two prospects (either sure or uncertain) facing the subject, and let him consider b better than a. Consider the following classes of prospects:

(1) a and b

(2) all lotteries that promise a or b

(3) all prospects that are not better than b and not worse than a. [Obviously (1) and possibly (2) are included in (3).]

(4) all prospects that are better than b

(5) all prospects that are worse than a.

9.3. The utilities in class (1) will be assigned arbitrarily, except for the condition that a is worse than b. We put $u(a)=0$, $u(b)=1$. [If classes (4) and (5) did not exist, a would stand for "agony" and b for "bliss"!]

The utilities in class (2) will all lie between (and excluding) 0 and 1, for the following reason: Let a lottery c promise b if a possible event s happens, and a if it does not happen. Hence c is better than a. For, if you acquire c and s happens, you get something better than a; while if s does not happen you get a. Similarly, c is worse than b. For, if you acquire c and s happens you get b; but if s does not happen you have something worse than b. Hence $u(a) < u(c) < u(b)$ and therefore $0 < u(c) < 1$, i.e., $u(c)$ is some proper fraction. [Compare also 6.1.]

Moreover compare two lotteries in class (2): $c_1$ and $c_2$, where $c_1$ promises b if the event $s_1$ happens. Let $p_1$ be the probability of $s_1$ and suppose $p_1 < p_2$. Then $u(c_1) < u(c_2)$, for the following reason. The lottery $c_2$ promises b with larger probability than does lottery $c_2$; but with smaller probability than a direct offer of b. Therefore $c_2$ can be conceived of as a lottery that promises $c_1$ and b with certain probabilities.*) Hence, by a reasoning similar to the one made before, $c_2$ must be better than $c_1$ and worse than b.

It follows that you will rank the utilities of the various prospects in class (1) by assigning increasing proper fractions as the probability p of getting b increases.

It is permissible therefore to choose as the utility rank of a lottery the fraction p if the lottery promises b with probability p. We have thus scaled all prospects of class (2), and this scale fits with the boundary values $u(a)=0$ and $u(b)=1$, since a and b can themselves be called lotteries, with $p=0$ and $=1$ respectively.

Before proceeding to the remaining classes of prospects, let us satisfy ourselves that our scale, so far, has the desired property. Let a prospect x be a lottery promising prospects $c_1, \ldots, c_n$ with probabilities $\pi_1, \ldots, \pi_n$. Show that

(9.B) $$u(x) = \Sigma\ \pi_i\ u(c_i).$$

To prove (9.B), we replace u(x) by the probability with which the lottery x promises b. This probability is compounded from $p_1, \ldots, p_2$, where $p_i$ is the probability with which $c_i$ promises b. Hence $u(x) = \Sigma\ \pi_i\ p_i$. But we have seen that $u(c_i)=p_i$. Hence (9.B) is true.

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*) Let these probabilities be q and 1-q, respectively. q is easy to find (though it is not really necessary for our purpose) by posing

$$q \cdot p_1 + (1-q) \cdot 1 = p_2,$$

since $p_2$, the chance of obtaining b, must be equal to the chance (in the new lottery) of getting it by virtue of having gotten $c_1$, plus the chance of getting it directly. We have $q = \frac{1-p_2}{1-p_1}$.

9.4. Consider now the class (3) of prospects. It includes classes (1) and (2), but it includes more. We have covered those of its members that are lotteries promising a or b, and were able to assign to each of these lotteries a utility p equal to the probability with which that lottery promises b. Of course, p ranges from 0 (the utility of a itself) to 1 (the utility of b itself). Now the utility of any member of class (3) must lie between 0 and 1 since it consists of prospects that are not better than b and not worse than a. Hence, any member of class (3) —say, $d_i$— that is not a lottery promising a or b has a utility that is equal to that of one of those lotteries —say, $c_i$— i.e., to some p $0 \leq p \leq 1$. If we now form a lottery y that promises various members $d_1, \ldots, d_n$ of class (3) with probabilities $\pi_1, \ldots, \pi_n$, then y has the same utility as the lottery x considered in (9.B). For, the event $s_1$ (with probability $\pi_1$) will give the subject the prospect $d_1$ if he had chosen y; and the prospect $c_1$ if he had chosen x. And so for $s_2, \ldots, s_n$. But since we have seen each $d_i$ have the same utility as the corresponding $c_i$ (this utility being equal to the probability with which $c_i$ promises b), it follows that the subject is indifferent between y and x. Thus (9.B) is extended to all members of the class (3), since u(y) can replace u(x) and each $u(d_i)$ can replace $u(c_i)$.

9.5. This would complete the proof of the Bernoulli Norm if we could assume that there exists for each subject a worst and a best prospect, "agony" and "bliss" (a and b). If this is not the case, i.e., utility is not bounded and classes (4) and (5) always exist, equation (9.B) still holds good. To show this, it suffices to pick the worst and the best prospect — say a' and b' — among those composing the particular lottery x [i.e., a' and b' will be among the $c_1, \ldots, c_n$ in (9.B)], and use a' and b' in the same way in which a and b were treated previously; i.e., create a new utility scale — say u' — with $u'(a')=0$, $u'(b')=1$, $u'(a)=p<q=u'(b)$, where p and q are certain probabilities: a and b having, respectively, the same utilities as certain two lotteries, each promising a' or b'. We see by former reasoning, that in terms of this new scale, (9.B) will be valid:

$$(9.C) \qquad u'(x) = \Sigma\ \pi_i\ u'(c_i).$$

But this latter equation remains valid also if the function u' is replaced by any linear transform of it, $\alpha u' + \beta$ (this can be easily verified by substitution). Now, the functions u' and u are, in fact, linear transforms of each other, with (for any c) $u'(c)=(q-p)\ u(c)+p$ [i.e., the scale u' is obtained from the scale u by shifting the origin by p and multiplying the utility unit by q-p]. Hence the validity of (9.C) entails the validity of (9.B) for any prospects,*) i.e., the validity of (9.A).

## X. HABITS vs. BERNOULLI NORM.

### 10.1. Ascertaining subjective probabilities. A distortion function.

In what follows, we assume that the experimenter has satisfied himself that the subject has learned to assign the events a subjective probability equal to that assigned them by the experimenter: that is, the latter probability satisfies the consistency norms given in Section VIII. If this is not the case, an unknown "distortion function" has to be introduced (and estimated from data, along with other functions involved in the hypothesis), which would give the subjective probability distribution of a given set of events as a function of its objective probability distribution.

*) With $u(a)=0$ and $u(b)=1$, the utility $u(e)$ of a member of class (4) and the utility $u(f)$ of a member of class (5) are easily shown to be

$$u(e)=\frac{1}{q}>1;\ u(f)=1-\frac{1}{r}<0,$$

where q and r are probabilities defined as follows: the subject is indifferent between b and a lottery promising e or a with respective probabilities q and 1-q; and he is indifferent between a and a lottery promising f or b with respective probabilities r and 1-r. It is easily verified that with these definitions,

$$u(e)\cdot q + u(a)\cdot(1-q) = u(b) \text{ and}$$
$$u(f)\cdot r + u(b)\cdot(1-r) = u(b).$$

10.2. Prospects not involving quantities of goods.

The Bernoulli Norm imposes restrictions on Models A, B, C of Section IV. For, in addition to the distribution F, one introduces the utility function u set of subjective probabilities $a_1, \ldots, a_n$. Or if only the objective probabilities are given, one introduces a function v on the set of subjective probabilities $a_1, \ldots, a_n$ where $v(a_i) = u(g(a_i))$ and g is the distortion function.

We shall neglect the distortion function for simplicity, thus assuming that the subject has learned the properties of the random device used. Note that in any case it might have been necessary to restrict the distortion function quite severly, e.g., in order to preserve that the sum of subjective probabilities be 1, etc.

Model A can then have, for example, the following form: There exists a n-variate distribution $F(x_1, \ldots, x_n)$ and a numerical function u on the set of prospects $a_1, \ldots, a_n$ such that

1) the probability (computable from F) that $x_h \geqslant x_i \geqslant x_j \geqslant \ldots$ equals the probability that the subject be ranking the set $a_1, \ldots, a_n$ by $a_h \; R \; a_i \; R \; a_j \; R \ldots$ ;
2) the marginal median of $x_i$ (computable from F) equals $u(a_i)$.

Clearly these restrictions may exclude the specification that F be normal, as in Model C.

Another specification might be considered, replacing 2) by

2') The probability (computable from F) that $x_i \geqslant x_j$ is a monotone non-decreasing function of the difference $d_{ij} = u(a_i) - u(a_j)$; it is equal $\frac{1}{2}$ when $d_{ij} = 0$. [Compare 4.2.2.]

10.3. Prospects involving quantities of goods.

As shown in 4.2.4, introducing quantities of goods implies further additional restrictions (incompatible with normality of F), since the probability is 1 that a large quantity will be preferred to a small one.

For example, suppose all prospects used in an experiment are lotteries, promising z dollars with probability p, and zero dollars with probability 1-p. Denote such a prospect by (z, p), and its utility by u(z, p). For simplicity, we shall disregard distortions. The last version of our Model of the preceding section becomes [compare 4.2.2 above]: For any real numbers z and any p, $0 \leq p \leq 1$ there exists a function u(z, p) and a univariate distribution G(z) such that

(1) $u(z, p) = p \cdot u(z, 1)$

(2) $G[u(z, p) - u(z', p')]$ equals the probability that the subject chose (z, p) in preference to (z', p').

(3) $G(0) = \frac{1}{2}$

(4) $G[u(z, p) - u(z', p')] = \begin{cases} 1 \text{ if } z > z',\ p \geq p' \text{ or } z \geq z',\ p > p' \\ 0 \text{ if } z < z',\ p \leq p' \text{ or } z \leq z',\ p < p' \end{cases}$

Clearly G — unlike in 4.2.2 — cannot be a normal distribution.

The experiments of Mosteller, Nogee, and Edwards provide data to test this model and to estimate the functions u and G.

For example, supposing that an infinitely large sample has been obtained, select all those pairs [(z, p), (z', p')] for which the frequency of preferring the first prospect to the second is the same. This gives by (2) prospect pairs with equal utility-differences, and thus enables to establish a utility scale. Since the sample was infinite, the Bernoulli Norm [assumption (1)] can be verified immediately. With a finite sample, of course, the inconsistencies can be "fitted out," at the price of possibly making the estimates of u and G subject to sampling errors so large as to make them useless for prediction.