COWLES COMMISSION DISCUSSION PAPER: ECONOMICS NO. 2096

Scheduling On A Single Railway Line So As To Minimize Accumulation Delay

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January 21, 1954

The study of the capacity of freight-carrying railroad systems can be approached as the problem of defining and determining the set of alternative transportation programs that can be carried out on a given network, with a given system of classification yards, and with a given amount of rolling stock, switch engines, line-haul engines, etc. One is particularly interested in the efficiency frontier of this set, that is, the set of alternative maximal programs that could be carried out by a given system.

As a first step toward exploring this efficiency frontier, this note discusses a highly simplified subproblem, that of determining the minimum amount of rolling stock required to carry out a given program on a network consisting of a single railway line with n terminals. The problem is further simplified in that classification delays to cars are disregarded (or treated as constant both as to time and as to place where incurred), and the main attention is given to scheduling the trains so as to minimize

* I am indebted to C. B. McGuire for valuable comments.

This paper is the result of research undertaken under contract between the Cowles Commission for Research in Economics and the RAND Corporation.

time spent waiting by cars until a train leaves.

The problem is discussed in two steps, one of choosing the structure of the schedule (as defined below), and one of choosing best departure times of trains in a schedule of given structure. Only for the second problem is a solution indicated.

Consider a single line with a set of yards (or terminals) numbered consecutively $i = 1, 2, \ldots, n$. Let the travel time d_{ij} of a train from i to j be constant and additive

$$(1) \qquad d_{ik} + d_{kj} = d_{ij}, \qquad i < k < j .$$

If we denote $d_{i,i+1}$ by d_i we obviously have

$$(2) \qquad d_{ij} = \sum_{k=i}^{k=j-1} d_k .$$

As said already, we disregard classification time and cost, and consider a stationary program of car flows x_{ij} all in the same direction ($x_{ij} = 0$ for $j \leqq i$).

A train is defined as an opportunity to ship cars from any yard i of a set $i_o \leqq i < j_o$ to any yard j of the set $i < j \leqq j_o$ at an incorporation time t_i given by

$$(3) \qquad t_i = t_{i_o} + \sum_{k=i_o}^{k=i-1} d_k$$

and with arrival time

$$(4) \qquad t_j = t_i + \sum_{k=i}^{k=j-1} d_k .$$

The set of integers $(i, i_o \leq i \leq j_o)$ is called the run of the train and is denoted (i_o, j_o); the time t_{i_o} is called its departure time. Our assumption (1) makes it possible to simplify the time coordinate by referring all times to the first yard of the system (called the reference yard)

$$(5) \qquad u = t_1 = t_i - \sum_{k=1}^{i-1} d_i \ ,$$

irrespective of whether $i_o = 1$ or $i_o > 1$. The time coordinate u will be called "referred time," or simply "time." A train is now fully described by its (referred) timing $u = t_1$ and its run (i_o, j_o).

By a periodic schedule we mean a sequence of trains u_r, (i_r, j_r) where $r = 1, 2, \ldots, R$ and

$$(6) \qquad 0 = u_1 \leq u_2 \leq \ldots\ldots \leq u_R \leq 1,$$

and where

$$(7) \qquad u_r = u_{r+1} \text{ implies } j_r < i_{r+1} \ .$$

This schedule is supposed to apply indefinitely, with

$$(8) \qquad u_{r+mR} = u_r + m, \ (i_{r+mR}, j_{r+mR}) = (i_r, j_r), \quad m = 1, 2, \ldots,$$

i.e., with periodicity 1. The subscript r is called the train number.

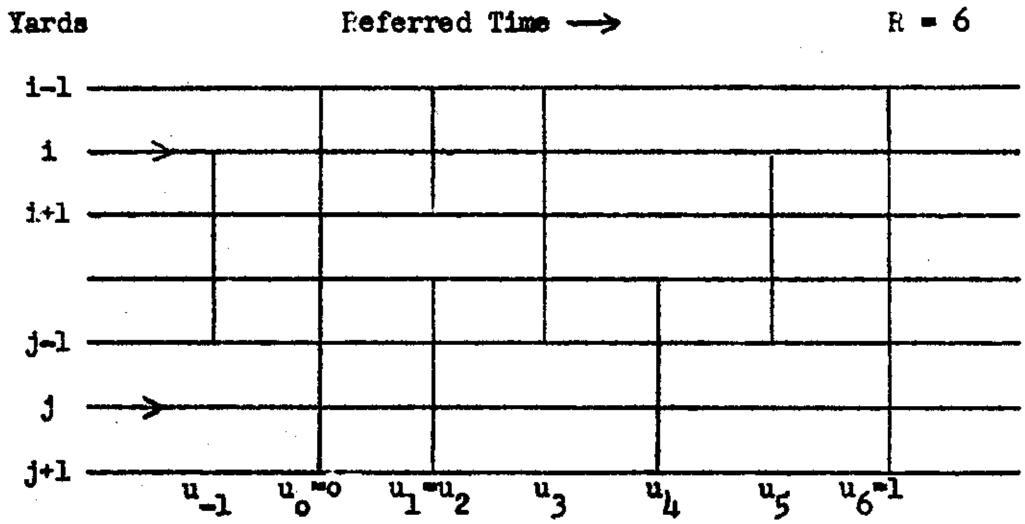
We shall now indicate more precisely how we split the choice of an optimum schedule in two sub-problems. We first choose the structure of the schedule, i.e. the sequence (i_r, j_r) of runs, and then we choose the timing u_r of each train subject to (6) and (7). The reason is that the composition of each train (in the sense defined below) can be chosen independently of the timing, within the restrictions (6) and (7), if the structure of the schedule is given.

In this note we only study optimal timing for a given structure. It will be necessary later to compare optima for different structures. If the cost of moving trains along a given sigment $(i, i + 1)$ is a linear function of the length of the train, and if cost for a run differs from the sum of the costs for the segments that constitute the run only by an overhead saving independent of train lengths on segments, then the total hauling cost is a function only of the structure of the schedule, not of the timings u_r of the trains. For this reason we here disregard hauling cost. If cost for a segment depends quadratically on train length on that segment, the minimand remains quadratic, and the problem can be solved by a simple extension of the procedure described below.

Our present problem then is, for a given structure of the schedule, to minimize total accumulation delay sustained during a period (since travel times are given constants, not alterable by changes in schedule or allocation of cars to trains). A preliminary to this problem is to specify, for a given schedule, an allocation of cars to trains that minimizes accumulation delay. We need only consider sets of cars $(i, j; \underline{u}, \overline{u})$ defined by specifying the yards i of origination, j of destination, and the referred time points $\underline{u}$ and $\overline{u}$ between which the cars in question accumulated at yard i. The number of cars in such a set is

$$x_{ij} \cdot (\overline{u} - \underline{u}) \quad . \tag{9}$$

Let us consider first only the allocation to trains of all cars from a given origin i to a given destination j. The following diagram illustrates the problem. It refers to a schedule of six trains per period.



Consider (i,j)-cars accumulating at i after departure of train o at time $u_o = o$. Their first chance of arrival at j is at time u_4. The last chance of departure for that arrival time is with train 3 at time u_3. Hence the allocation of cars of the set $(i,j;\, u_o, u_3)$ must be such that they arrive at j at time u_3, but need not be specified beyond that for those cars for which more than one allocation meets this objective, because neither accumulation delay, nor, under present assumptions, haulage cost depends on it. The total accumulation time of cars in the set $(i,j;\, u_o, u_3)$ is, under such an allocation, x_{ij} multiplied by

$$\int_{u_o}^{u_3} (u_4 - u)du = \left[-\frac{1}{2}(u_4 - u)^2 \right]_{u_o}^{u_3} = \left[-\frac{1}{2} u_4^2 + u_4 u - \frac{1}{2} u^2 \right]_{u_o}^{u_3}$$

$$= u_4(u_3 - u_o) - \frac{1}{2}(u_3^2 - u_o^2) = (u_3 - u_o)\,(u_4 - \frac{u_3 + u_o}{2})$$

$$= (u_3 - u_o)\,(u_4 - u_3) + \frac{1}{2}(u_3 - u_o)^2$$

The next set, $(i,j;\, u_3 u_6)$ has earliest arrival time at u_6, with accumulation time x_{ij} multiplied into

$$\int_{u_3}^{u_6} (u_6 - u)du = \frac{1}{2}(u_6 - u_3)^2 .$$

None of the other times in the schedule $(u_1 = u_2, u_5)$ enter into accumulation time for (i,j)-traffic, because the removal of any of the trains in question does not cause an (i,j)-car either to arrive at j later or requires it to leave from i earlier.

This consideration suggests the following distinctions. The pair of yards (i,j) defines two sequences u_s^{ij} and w_s^{ij}, $s = 1, \ldots, S_{ij}$, such that (symmetrically, and omitting sub- and super-scripts i,j)

(10a) v_s is the latest departure time from i permitting arrival at j at time w_s.

(10b) w_s is the earliest arrival time at j possible for cars leaving i at time v_s.

(10c) v_{s-1} is the latest departure time from i permitting arrival at j at a time prior to w_s.

(10d) w_{s+1} is the earliest arrival time at j possible for a car leaving i at a time later than v_s.

(10e) $$v_s = v_{S+s} + 1, \qquad\qquad w_s = w_{S+s} + 1$$

The existence of such sequences follows from the method of construction illustrated above. In our example we have

(11) $$\begin{matrix} v_1 = u_o, & v_2 = u_3, & v_3 = u_6, \\ w_1 = u_o, & w_2 = u_4, & w_3 = u_6, \end{matrix} \qquad S = 2$$

Accumulation time for (i,j)-traffic over one period of the schedule can now be written in general as

(12) $$A_{ij}x_{ij} = x_{ij} \cdot \sum_{s=1}^{S_{ij}} \left[\frac{1}{2}(v_s^{ij} - v_{s-1}^{ij})^2 + (v_s^{ij} - v_{s-1}^{ij})(w_s^{ij} - v_s^{ij})\right]$$

Hence total accumulation delay

(13) $$A = \sum_{\substack{i,j=1 \\ i<j}}^{n} A_{ij} \, x_{ij}$$

is a quadratic function of all train timings u_r.

We distinguish two classes of structures. In class I a minimum of A is reached in a single internal* point of the region U in the space of $u_1, \ldots, u_R$, defined by (6) and (7). In this case A is necessarily positive definite in the u_r, $r=1, \ldots, R$, and the minimizing schedule of the given structure is found by solving the linear equation system

(14) $$\frac{\partial A}{\partial u_r} = o, \qquad r=1, \ldots, R-1.$$

This equation system can in any particular case be set up by determining $\frac{\partial A_{ij}}{\partial u_r}$ separately for each traffic component (i,j). We have, omitting sub- and superscripts ij (except from A_{ij}),

(15) $$\frac{\partial A_{ij}}{\partial u_r} = \sum_{s=1}^{S} \frac{\partial A_{ij}}{\partial v_s} \frac{dv_s}{du_r} + \sum_{t=1}^{S} \frac{\partial A_{ij}}{\partial w_t} \frac{dw_t}{du_r} \quad ,$$

where $\frac{dv_s}{du_r}$ and $\frac{dw_t}{du_r}$ are o except for at most one value of s and/or t, i.e. the values (if any) such that

(16) $$v_s = u_r \qquad \text{and/or} \qquad w_t = u_r \quad .$$

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* A complication is created by condition (7), illustrated in our diagram by trains 1 and 2. The only reason for condition (7) is notational: it provides us with a unique numbering of trains. However, in minimizing A we should allow both u_1 and u_2 in our example to vary freely in the interval $u_o < u < u_3$. Only those restrictions (6) that govern the making or missing of connections, need to be insisted on. This means that essentially the structure is not a sequence of runs, but a set of runs in which there is a partial ordering that orders all runs containing any given yard in a sequence, such that the sequences precipitated by different yards are consistent. This seemed too complicated to burden the exposition with.

This rule together with

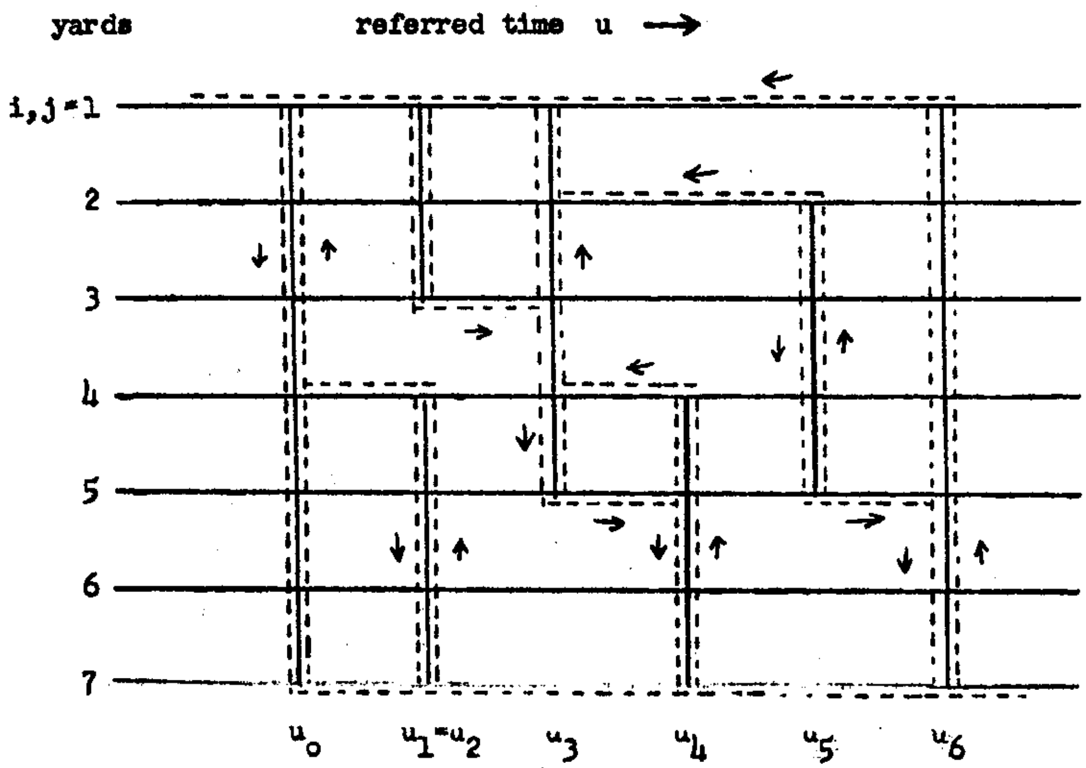
$$\frac{\partial A_{ij}}{\partial v_s} = (v_s - v_{s-1}) - (v_{s+1} - v_s) + (w_s - v_s) - (w_{s+1} - v_{s+1}) - (v_s - v_{s-1})$$

$$(17) \qquad = w_s - w_{s+1}$$

$$\frac{\partial A_{ij}}{\partial w_s} = v_s - v_{s-1}$$

completely defines $\frac{\partial A_{ij}}{\partial u_r}$.

The determination of the sequences v_s^{ij}, w_s^{ij} for each traffic component (i,j) is facilitated by a diagram which we illustrate from the above example.



Dotted lines immediately to the right of train lines are traced upward. Dotted lines immediately above yard lines are traced to the left. These lines can serve for the determination of the sequences v_s^{ij} and w_s^{ij} as follows. Let $i=2$, $j=6$ and consider the point $j=6$, $u=u_4$. Tracing dotted lines upward and to the left this leads to $u=u_3 = v_2^{26}$ as latest departure time from yard 2. There is no earlier time than u_4 at which yard 6 can be reached from yard 2 departure at $u=v_2^{26}$, hence $u_4 = w_2^{26}$.

The nearest earlier arrival time at 6, $u=u_2$, requires latest yard 2 departure at $u=u_o=v_1^{26}$. The earliest arrival at 6 from 2 departure at $u=v_1^{26}$ is at $u=u_o=w_1^{26}$. In this way both sequences can be derived from the upward and leftward dotted lines. Symmetrically, they can likewise be traced from dotted lines immediately to the left of trainlines, traced downward,

" " " below yard " , " to the right.

The quickest method is alternating use of both line systems. Pairs of points (i,u_r) and $(j,u_{r'})$ such that one can reach either from the other through the appropriate line system are such that $u_r=v_s$, $u_r=w_s$ for some s. Points not occurring in such a pair belong to neither sequence.

The same lines are of course used for each traffic component (i,j), but the resulting sequences generally depend on (i,j). The following table (13) applies to the above example. It lists for each (i,j) the numbers r and r' such that $u_r=v_s$, $u_{r'}=w_s$, in order of increasing s.

	j=2	3	4	5	6	7
i=1	r=0 1 3 6 r'=0 1 3 6	0 1 3 6 0 1 3 6	0 3 6 0 3 6	0 3 6 0 3 6	0 3 6 0 4 6	0 3 6 0 4 6
2		0 1 3 5 6 0 1 3 5 6	0 3 5 6 0 3 5 6	0 3 5 6 0 3 5 6	0 3 6 0 4 6	0 3 6 0 4 6
3			0 3 5 6 0 3 5 6	0 3 5 6 0 3 5 6	0 3 6 0 4 6	0 3 6 0 4 6
4				0 2 3 4 5 6 0 2 3 4 5 6	0 2 3 6 0 2 4 6	0 2 3 6 0 2 4 6
5					0 2 4 6 0 2 4 6	0 2 4 6 0 2 4 6
6						0 2 4 6 0 2 4 6

From tables (12) and (13) we can tabulate the coefficients of each u_q in $\frac{\partial A}{\partial u_r}$ as follows (leaving off r=r'=o which is already represented by r=r'=6).

(19) Table of coefficients of

	u_1	u_2	u_3	u_4	----
in $\frac{\partial A}{\partial u_1}$	$2(x_{12}+x_{13}+x_{23})$		$-(x_{12}+x_{13}+x_{23})$		-----
$\frac{\partial A}{\partial u_2}$		$2(x_{45}+x_{46}+x_{47}$ $+x_{56}+x_{57}+x_{67})$	$-x_{45}$	$-(x_{46}+x_{47}+x_{56}$ $+x_{57}+x_{67})$	-----
$\frac{\partial A}{\partial u_3}$	$-(x_{12}+x_{13}+x_{23})$	$-(x_{45}+x_{46}+x_{47})$	$2\sum_{\substack{i,j=1\\ i<j}}^{5} x_{ij}$	$\sum_{i=1}^{5}\sum_{j=6}^{7} x_{ij}$ $-x_{45}$	-----
$\frac{\partial A}{\partial u_4}$		$-(x_{46}+x_{47}+x_{56}$ $+x_{57}+x_{67})$	$\sum_{i=1}^{5}\sum_{j=6}^{7} x_{ij}$ $-x_{45}$	$2(x_{45}+x_{56}$ $+x_{57}+x_{67})$	-----
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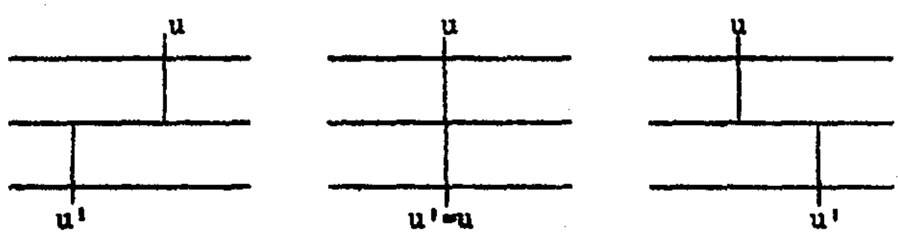
The table is structured as follows: For each occurrence of a given integer r in the first (v_s-)row of an (i,j)-cell, the amount x_{ij} is entered with a plus sign in the coefficient of $u_{r'}$, where r' is the integer occurring in the second (w_s-) row immediately below r, and with a minus sign in the coefficient of $u_{r''}$, where r'' is the integer to the right of r' in the second row. Similarly, for each occurrence of a given r' in the second row of an (i,j)-cell, the amount x_{ij} is entered with a plus sign in the coefficient of u_r, where r is the corresponding element in the first row, and with a minus sign in the element of $'r$, where $'r$ is the integer to the left of r. The values $r=o$ and $r=6$ are to be treated as identical.

If a structure is in class I (minimum accumulation delay in an interior point of U), it does not follow necessarily that that structure is efficient. Considering the three cases illustrated below



the accumulation time function makes a downward jump when u' reaches the value $u'=u$ from below, but is continuous for $u' \geqq u$ in a neighborhood of u. Probably haulage cost behaves in a similar manner, but its jump need not be by the same amount.

On the other hand, if a structure is in class II (minimum at a boundary point, corresponding to another structure), this is a clear indication that the structure in question is <u>not</u> efficient.

The next problem is to examine whether the present analysis gives a basis for analysis of the choice of an efficient structure. Additional data on haulage cost to determine the cost of alternative structures are needed for that problem.

McGuire has proposed in discussion to utilize a structure concept which contains only trains connecting adjacent yards, while (7) is weakened to allow $j_r = i_{r+1}$. The advantages of this procedure may depend on the computation method used in giving effect to the inequalities (6) and (7) in minimizing a quadratic function. An advantage may also lie in the straight forward generalization of this treatment of structure to schedules containing mainliners (trains that do not stop at all yards they pass).