COWLES COMMISSION DISCUSSION PAPER: ECONOMICS NO. 2095

On A Variational Problem in Non-Static Linear Activity Analysis*

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January 13, 1954

1. One may speak of non-static activity analysis if the allocation of resources is subject to limitations on both stock and flows. Thus if A, B are the matrices of flow and stock input coefficients, respectively and c is the flow of resources currently available, then the choice of activity levels x is constrained by inequalities

$$A\,x + B\,\dot{x} \leqq c$$

$$x \geqq 0.$$

B x, the vector of stocks required to sustain activities of level x, may be regarded as (physical) capital. The technological assumptions here made to correspond to those of dynamic Leontief models if technological choice is allowed for. One interpretation is that the B x represents quantities of intermediate commodities "in the pipe-line".

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* Research undertaken under contract between the Cowles Commission for Research in Economics and the RAND Corporation.

The object of maximization are the outputs at various times at values discounted to an initial time point, plus, possibly, the value of capital stocks at the end of the period under consideration,

$$\int_{0}^{T} f' x \, d t \qquad + \qquad f' x (t).$$

While capital theory has emphasized the significance of states for which the activity levels are exponentially expanding (processes of balanced growth) including the case of constancy (the stationary state), the ways in which transitions come about from one set of positive activities to another are of interest, too. In order to study this phenomenon by means of activity analysis, a generalization of its basic theorem on implicit pricing is desirable. The present paper formulates the mathematical problem and solves it for a class of cases, corresponding somewhat to the non-degenerate problems of discrete activity analysis.

The present type of a linear variational problem has arisen also in the so-called theory of dynamic programming [1]. Our viewpoint stresses the existence of non-negative Lagrangean parameters rather than employing the elegant tool of functional equations, which would seem further removed from economic intuition. The method of Lagrangean multipliers in variational problems involving differential inequalities as constraints has been studies previously by Bolza [2] in a pioneering paper and again with emphasis on sufficient conditions by Valentine [3]. While these authors do not restrict themselves to linear problems, they impose restrictions on the rank of the constraints system: the matrix B must have a rank equal to its number of rows. Since this would render our economic (as well as the mathematical) problem quite trivial, it seems best to start from scratch for a fuller analysis of the particular linear problem on hand.

It seems useful to embody the inequalities

$$x \geqq 0$$

in the set $Ax + B\dot{x} \leqq c$ in order to obtain the simpler formulation that follows.

<u>Problem</u> (1) $f' \, x(t) + \int_0^T g' \, x \, dt = \underset{x}{\text{Max}}$

subject to (1a) $Ax + B\dot{x} \leqq c$

(1b) $B\Delta \, x \leqq 0$

(1c) $Dx\,(0) \leqq e$

(1d) $\dot{x}$ the directional derivative of x with respect to the single valued parameter t, is piecewise continuous

Here A, B are constant matrices of m rows and n columns each, $m \geqq n$,

g, c are vectors of n, m components respectively, and are piecewise continuous functions of t

e, f are given constant vectors, and

Δx denotes $x\,(t + 0) - x\,(t - 0)$.

The case we are able to handle mathematically is that in which the solutions have the following properties:

1) the set of t for which the "=" sign holds in any particular inequality consists of at most finitely many intervals (to the exclusion of isolated points).

2) on any interval a subset of equations among (1a) either states a contradiction or is linearly independent.[1/]

1/ It can be seen, that given A, B, c there always exists a $\bar{c}$ such that these conditions are satisfied with $A, B, \bar{c}$ and such that $(c - \bar{c})(c - \bar{c}) < \epsilon$ for all t.

For purposes of this paper a differential system

$$(1') \qquad Px + Q\dot{x} = a$$
$$Qx(0) = Qb$$

will be called linearly independent if there exists one and only one solution for every piecewise continuous $a(t)$ and every b.

To see the significance of this statement, let $r \leq n$ be the rank of the matrix Q. There always exist two non-singular $n \times n$ matrices S, T such that

$$S'QT = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}, \text{ where } R \text{ is non-singular, and } r \times r.$$

Write correspondingly

$$S'PT = \begin{pmatrix} E & F \\ G & H \end{pmatrix} \qquad x = (y, z)$$

Then the equation system takes the form

$$Ey + Fz + R\dot{y} = k$$
$$Gy + Hz = l$$
$$Ry(0) = Rp$$

where $\begin{pmatrix} k \\ l \end{pmatrix} = S a$

$$\begin{pmatrix} p \\ q \end{pmatrix} = T^{-1} b$$

By linear independence we mean that there exists one and only one solution y, z for each k, l, p. Uniqueness of the solution implies that H be non-singular. Then the equation system reduces to the standard form

$$\dot{y} + R^{-1}[E - FH^{-1}G]\,y = R^{-1}(k - FH^{-1}l)$$
$$y(0) = p$$

Since this clearly has a solution for every k, l, p, it follows that nonsingularity of H is also sufficient for system (1') to be linearly independent. We note for later use, that if (1') is a linearly independent system, so is the adjoint system

$$P'x - Q'\dot{x} = d$$

$$Q'x(T) = Q'e$$

For then the matrices T', S' replace S, T respectively and it is necessary and sufficient for linear independence that H' be non-singular.

It is conjectured that the theorem to be proved is valid also without the two assumptions introduced before. We shall also assume at first that the problem solution (if it exists) is unique. This will be seen to imply no loss of generality.

2. Suppose then that x is a solution. Let $k = 1, \dots K$ denote the subintervals on which the sets of equations among (1a) for x are constant. These equation systems, to be written

$$(1a') \qquad A_k x + B_k \dot{x} = c_k \qquad t_{k-1} \leqq t \leqq t_k \quad k = 1, \dots K$$

must be complete and linearly independent everywhere by previous assumptions. This implies in particular that the matrices A_k, B_k be $n \times n$.
For brevity (1a') will be written

$$L x = \bar{c}$$

The subset of initial condition (1c) in which the "=" sign is assumed for the solution x shall be written $\bar{D} x = \bar{e}$. Finally let $\bar{B}_k$ denote the matrix of rows for which the = sign is taken on at $t = t_k$ in constraint (1b).

That x is a maximizer now implies

$$(2) \qquad f' u(t) + \int_0^T g' u \, d t \leqq 0$$

for all "variations" u with

(2a) $A_k u + B_k \dot{u} \leqq 0$

(2b) $\bar{B}_k \Delta u \leqq 0$

(2c) $\bar{D} \; u(0) \leqq 0$

(2d) $u, \dot{u}$ piecewise continuous.

In order to eliminate the constraints (2a) consider a system of equations adjoint to (2a)

(3) $A'_k v - B'_k \dot{v} = g \qquad t_{k-1} \leqq t \leqq t_k \qquad k = 1, \ldots K$

This system will be written $L^* v = g$. Since (3) also is a linearly independent system (p. 5) there exists a solution on each interval k for every piecewise continuous g and for every terminal condition of the form

(3a) $B'_k v(t_k - 0) = B'_k h_k$ with arbitrary h_k

We may substitute now for g' in (2)

$$f' u(T) + \int_0^T (L^* v)' u \, d t \leqq 0$$

and through integration by parts obtain

$$f' u(T) + \int_0^T v' L u \, d t - \sum_k v' B_k u \Big/_{t_{k-1}}^{t_k} \leqq 0$$

Writing $y(t_k + 0) - y(t_k - 0) = \Delta_k y$ and

$$B_k = B(t) \qquad t_{k-1} \leqq t \leqq t_k$$

this becomes

(4) $$\int_0^T v' L u \, d t + [f' - v'(T) B_k] \; u(T) + v(0)' B_1 u(0) + \sum_{k=1}^{K-1} \Delta_k (v' B u) \leqq 0.$$

As a first choice let

$$u(t_k) = 0 \qquad k = 1, \ldots K - 1 \qquad \text{and}$$
$$L\,u = 0 \qquad \text{all } t.$$

Then, since $L\,u$ is a linearly independent system, $u = 0$ for all $0 \leqq t < T$. Applying (2) and (2b) at $t = T$, we have

$$f'\,u(T) \leqq 0 \quad \text{for all } u(T) \text{ with}$$
$$B_K\,u(T) \leqq 0$$

According to the Lemma of Farkas [4] there exists an $S \geqq 0$ such that $f' = S'B_K$. If we now choose h_K in (3a) equal to S we have

$$v(T - 0)'\,B_K = S'B_K = f'$$

and the second term of (4) disappears.

As a next choice let

$$u(t_k) = 0 \quad k = 0, \ldots K-2$$
$$u(t_{K-1} - 0) = 0$$
$$L\,u = 0 \qquad \text{all } t.$$

Then (2) says that

$$v'(t_{K-1})\,B_K\,u(t_{K-1}) \leqq 0 \quad \text{for all } u(t_{K-1}) \text{ with}$$
$$\tilde{B}_{K-1}\,u(t_{K-1}) \leqq 0.$$

Again the Farkas Lemma ensures the existence of a non-negative vector W with

$$v'(t_{K-1})\,B_K = W'\,\tilde{B}_{K-1}$$

Since $\tilde{B}_{K-1}$ denotes the rows common to B_K and B_{K-1} we may choose h_{K-1} of (3a) equal to W. Then

$$v'(t_{K-1} - 0)\,B_{K-1} = h'_{K-1}\,B_{K-1} = W'\,\tilde{B}_{K-1} = v'(t_{K-1})\,B_K.$$

In other words $v' B_{K-1}$ is a continuous extension of $v' B_K$ at $t = t_{K-1}$. Moreover by definition of $\tilde{B}_{K-1}$

$$v' B_{K-1} x = v' B_K x \quad \text{at} \quad t = t_{K-1}$$

These considerations may be repeated for $t = t_{K-2}, \ldots, t_1$ with the result that

$$v' (t_k - 0) B_{k-1} = v' (t_k) B_k \quad \text{and}$$

$$v' (t_k - 0) B_{k-1} x (t_k - 0) = v' (t_k) B_k x (t_k) \quad \text{for} \quad k = 1, \ldots K - 1.$$

In the third place let u be chosen continuous, so that $v' L u$ is continuous and the last term of (4) drops out. In addition choose $L u = 0$. Then (4) states that

$$v' (0) B_1 u (0) \leqq 0 \quad \text{if} \quad \bar{D} u (0) \leqq 0$$

Again we conclude from the Farkas Lemma that

$$v' (0) B_1 = Z' \bar{D} \quad \text{where} \quad Z \geqq 0.$$

Finally let $u (0) = 0$ and u be continuous. Since L represents a linearly independent system there exists a piecewise continuously differentiable solution to $L u = \not{l}$ for every piecewise continuous $\not{l}$. Thus if

$$\not{l} (t) = \min (0 , v (t))$$

u satisfies all constraints including $L u \leqq 0$. But

$$\int_0^T v' L u \, d t = \int_{v>0} v' v \, d t > 0 \quad \text{contradicting (4)}$$

unless v is non-negative almost everywhere. Since v is piecewise continuous it then is non-negative everywhere. Fill in zero elements to render the v into vectors of n elements, to be called λ. Then by construction $\lambda' \cdot [c - L x] = 0$ for all t.

We have thus proved the "necessity" part of the following <u>Lemma</u>. Under the stated assumption a necessary condition for x to be a solution of the problem is the existence of a vector λ with piecewise continuous directional derivatives and of a vector μ such that

(6a) $A' \lambda - B' \dot{\lambda} = g$

(6b) $\lambda' (A x + B \dot{x} - c) = 0$

(6c) $B' \lambda (T) = f$

(6d) $B' \lambda (0) = D' \mu$

(6e) $\mu' (D x (0) - e) = 0$

(6f) $B' \lambda$ continuous, $\lambda' B x$ continuous

(6g) $\lambda \geqq 0, \mu \geqq 0$

3. Conversely suppose now that the condition (1) and 6 are satisfied by some x and λ. Then for every z satisfying (1a)...(1d)

$$f' z(T) + \int_0^T g' z \, d t = f' z (t) + \int_0^T (\lambda' A - \dot{\lambda} B) z \, d t$$

$$= [f' - \lambda' (T) B] z (T) + \lambda'(0) B z (0)$$

$$+ \int_0^T \lambda' [A z + B \dot{z}] \, d t + \sum_k \Delta_k (\lambda' B z)$$

$$\leqq \mu' D z (0) + \int_0^T \lambda' c + \sum_k \lambda' B \Delta_k z \leqq \mu' e + \int_0^T \lambda' c$$

and "=" holds if $z = x$. Therefore (1) and (6) are also sufficient for x to be a maximizer.

As a corollary we note that the Lagrangean function

$$f' y (T) + \mu' [e - Dy (0)] + \int_0^t (g' y + \lambda' [c - Ay - B\dot{y}]) \, d t$$

is constant for all continuous y with piecewise continuous directional derivatives. Suppose now the problem solution is not unique. Then the set of constraints taken on in (1a) consists of less than n equations on some subintervals. Changing the c components in the remaining inequalities does not affect the solution provided that the constraints remain compatible with the particular solution. In this way we may complete everywhere the equation systems (1a') to n linearly independent equations. We then apply the lemma and obtain a set of λ. The value of the maximum now is easily calculated

$$= \int_0^t \lambda' [A x + B \dot{x}] \, d t + \mu' C x (0)$$

But since all those rows of the constraints that did not assume the "=" sign in the original problem may be given any continuous value without affecting the maximum, it follows that the respective components of λ and μ must be zero. This proves the lemma for the case of non-unique solution.

On the other hand it is seen that we are free to choose the solution such that it actually assumes the "=" sign in n constraints everywhere. To summarize, under the assumptions of this paper, a certain solution x of the problem can be characterized by the following necessary and sufficient conditions:

There exists a partitioning of 0 T into subintervals k, $k=1,\ldots K$, such that on each of these x is the solution of a complete system of differential equations, equal in number to that of components of x,

$$A_k x + B_k \dot{x} = c_k$$

where A_k and B_k are composed of a subset of rows of A, B respectively and c_k consists of the corresponding components of c. Let $\bar{B}_k$ denote the rows common to B_k and B_{k+1}. Then $\bar{B}_k x$ remains continuous at the transition from interval k to $k+1$.

The effective initial conditions $\bar{D}\, x\,(0) = \bar{e}$ are such that they are compatible with the differential equation

$$A_1 x + B_1 \dot{x} = c_1$$

The intervals k and the matrices A_k, B_k are fixed by means of the following requirements: The adjoint system $A'_k v - B_k' \dot{v} = g$ has non-negative solutions v such that $v' \bar{B}_k$ remains continuous, all other components of $v' B_k$ and $v' B_{k+1}$ vanish at the interval endpoint t_k, and the boundary conditions hold

$$v\,(T)' B_k = f'$$

$$v\,(0)'\, B_1 = \mu' D$$ where μ is some non-negative vector whose components vanish unless the corresponding constraints in $Dx \leq e$ assume the "=" sign.

In economic applications the constraints (1a) usually include a set $x \geqq 0$. In this case the Lagrangean Multiplier Rule may be stated in more symmetric form

<u>Theorem</u> $$f' \, x(t) + \int_0^T g' \, x \, d\, t = \max_x$$

subject to
$$A\,x + B\,\dot{x} \leqq c$$
$$B\,\Delta\,x \leqq 0$$
$$D\,x(0) \leqq e$$
$$x \geqq 0$$

$x, \dot{x}$ piecewise continuous

if and only if there

exists λ, μ such that

$$\lambda' A - \lambda' B \geqq g'$$

$\lambda' B$ continuous

$$\lambda(0)' \, B = \mu' D$$

$$\lambda(T)' \, B = f'$$

$$\geqq 0$$

$\lambda, \dot{\lambda}$ piecewise continuous, and

$$\lambda' (A\,x + B\,\dot{x} - c) = 0$$

$$(\lambda' A - \dot{\lambda}' B - g)\, x = 0$$

Corollary: $$f' \, x(T) + \int_0^T (g' x + \lambda' [c - A\,x - B\,\dot{x}])\, d\,t + \mu' [e - D\,x] = \max_{x \geqq 0} = \min_{\lambda \geqq 0}$$

If, as at the beginning, we let

x	represent	activity levels
A		flow input coefficients

B	represent	stock (capital) input coefficients
g		values of flow of net outputs discounted back to the present
f		values of stock items at end of period discounted to the present
e		initial availability of stock (then C = B)
c		current availabilities of flow inputs,

the theorem asserts that, as in static activity analysis, the problem solution implies an imputation of shadow pieces, discounted to the present, to all commodities such that inefficient activities are penalized with (book) losses and efficient ones yield zero (book) profits. As usual prices are zero whenever the corresponding availability limit is not reached. A new feature is the continuity properties of prices and stocks: While individual prices may jump discontinuously (λ is piecewise continuous) the per unit values of stocks $B'\lambda$ behave continuously. Similarly although stocks may decrease in discontinuous fashion ($\Delta B x \leq 0$), the value of existing stocks $\lambda' B x$ is continuous, jumps in stock levels being restricted to those stocks that have zero prices at that time. In other words, no speculative gains or losses may be had by holding the stocks required for an efficient allocation of resources. This is, of course, but a consequence of the absence of uncertainty in our allocation model.

With the apparatus of non-static activity analysis insight may be obtained into the stationary state. For instance it can be shown that if c is a constant vector and g of the form $g_0 e^{-rt}$ and if a solution exists then x becomes stationary for sufficiently large t, and the stationary state attained depends on the discount rate r. In particular the value of output in the stationary state in terms of the undiscounted

g_o in a non-increasing function of r, provided $B \geqq 0$. We have here the rudiments of an interest theory for linear models of balanced or unbalanced growth, which we hope to develop more fully in a later paper.

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