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The Firm as a Team[1]

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I. FORMAL CONCEPTS

The firm is looked at as a "team," i.e., a group of decision makers with a common goal. Each decision maker bases his decisions upon information about external, random variables; the nature of information is, in general, different for different members of the team and depends on the "network."

FORMAL CONCEPTS		EXAMPLE A: Production of a Single Commodity	EXAMPLE B: Speculative Shipping of a Single Commodity
decision variables:	vector a	levels of inputs a_i $(i=1,\ldots,n)$	quantities shipped from city i to city j a_{ij} $(0 \leq a_{ij};\ \sum_j a_{ij} = 1)$
external random variables:	vector x	prices of inputs x_i (price of output = 1)	predicted price in city i x_i
gross payoff	u(a,x)	profit: $f(a_1,\ldots,a_n) - \Sigma x_i a_i$ where f is production function	profit: $\sum_i \sum_j a_{ij}(x_j - x_i - t_{ij})$ where t_{ij} is transportation cost from i to j

Information structure S, specifies what information (usually a vector) $y^{(i)} = S_i(x)$ each decision variable a_i is based upon.

Decision rule $\alpha = (\alpha_1,\ldots,\alpha_n)$, determines each decision variable as a function $a_i = \alpha_i(y^{(i)})$, of the information $y^{(i)}$.

Expected gross payoff $= U(\alpha,S) = \mathcal{E}\{u(\alpha(y), x)\}$

Network, specifies how the information structure is to be achieved.

Cost of structure = C(S) = the cost of the cheapest network corresponding to S.

Expected net payoff $= U(\alpha, S) - C(S)$.

Interaction between decisions: $\partial^2 u / \partial a_i\, \partial a_j,\ i \neq j$

Interaction between decision and nature: $\partial^2 u / \partial a_i\, \partial x_j$ (assuming u differentiable).

The problem of the team is to choose S and α so as to maximize $U(\alpha, S)-C(S)$. In the absence of specific assumptions about costs one may discuss the problem of choosing, for each S, the best α, and of finding the advantage of one information structure over another.

Suppose S is fixed, and let $\hat{\alpha} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_n)$ denote the best decision rule corresponding to S. This best rule must satisfy the following set of n conditions:

(1) For any fixed i and $y^{(i)}$, and for $\alpha_j = \hat{\alpha}_j$ $(j \neq i)$, $\hat{\alpha}_i(y^{(i)})$ must be equal to an a_i which maximizes the conditional expectation $\mathcal{E}\{u(\alpha, x) | y^{(i)}\}$.

Note that for given functions α_j $(j \neq i)$ and given $y^{(i)}$ this conditional expectation is a function of a_i alone. The above conditions are in general necessary but not sufficient. In the examples considered here they are sufficient, too.

In the following examples the information structure will always be such that the information $y^{(i)}$ consists of certain of the components of x. Such a structure can be represented by a matrix $((s_{ij}))$ where $s_{ij} = 1$ or 0 according as a_i does or does not depend upon x_j.

II. <u>EXAMPLE A. Production of a Single Commodity.</u>

$a = \{a_i\}$, levels of inputs $(i = 1, \ldots, n)$.

$x_i = x_i(a)$, prices of inputs, allowing the possibility of imperfect markets; price of output = 1.

$f(a)$, production function.

$c(a,x) = \Sigma\, a_i\, x_i(a)$, total cost of inputs.

$f(a)-c(a,x)$, profit.

$\alpha_i(y^{(i)})$, decision rule determining a_i, i.e., $a_i = \alpha_i(y^{(i)})$

$y^{(i)}$, vector of prices on which a_i depends.

Best rule must satisfy [see condition (1)]

$$(2) \qquad \mathcal{E}\left\{\frac{\partial}{\partial a_i} f(\alpha(y))\middle| y^{(i)}\right\} = \mathcal{E}\left\{\frac{\partial}{\partial a_i} c(\alpha(y),x)\middle| y^{(i)}\right\}$$

i.e., conditional expected marginal product equals conditional expected marginal cost. [Compare with the condition under certainty, "marginal product equals marginal cost."]

Under competitive conditions, the x_i are independent of a and

$$(2a) \qquad \mathcal{E}\left\{\frac{\partial}{\partial a_i} f(\alpha(y))\middle| y^{(i)}\right\} = \mathcal{E}\left\{x_i \middle| y^{(i)}\right\}$$

Special case: 2 inputs

prices normally distributed

quadratic production function

By suitable choice of units of inputs, profit can be written

$$u(a,x) = \frac{1}{4}\left[u_o - (a_1^2 - 2q\, a_1\, a_2 + a_2^2) - (a_1\, x_1 + a_2\, x_2)\right], \ (q^2 < 1)$$

with u_o a constant; x_i, deviation of price from mean (hence $\mathcal{E}x_i = 0$); a_i, deviation from level which is best when each $x_i = 0$. Let $\mathcal{E}x_1^2 \leq \mathcal{E}x_2^2$, and choose money unit to make $\mathcal{E}x_1^2 = 1$. Write $\mathcal{E}x_2^2 = \tau^2$ and $\mathcal{E}x_1\, x_2 = \rho\tau$.

The expected gross payoff of the given structure will thus depend on three parameters:

q, $= 2 \frac{\partial^2 u}{\partial a_1 \partial a_2}$, a measure of interaction between the two inputs in production;

τ, a measure of the ratio of uncertainties about the two prices; and

ρ, coefficient of correlation between prices.

There are altogether $2^4 = 16$ structures. We can assume that the i-th price is always learned at a smaller cost by the person deciding about the i-th input than by the other person. Then it turns out that only nine structures have net expected payoffs that are not dominated (for all values of parameters) by those of some other structures. The gross expected payoffs of all 16 structures are given in Table I, where, for example, the column heading (10) means that the first input is decided on the basis of the knowledge of the first but not of the second price; row headings refer correspondingly to the second input. Cells corresponding to the non-admissible (dominated) structures are crossed out. The remaining, i.e. admissible, cells contain, in addition to the maximum expected gross payoff, symbols for the network that generates the given structure, in the following notation: a square, □ means "no observation," x means "observation," and arrow means the direction of communication. The left (right) place corresponds to the person deciding about the level of the first (second) input and possibly observing its price. Thus x→□ means: first price is observed by the first and communicated to the second person; second price not observed.

Table II gives the best rules of action for the 16 different structures, arranged in the same manner as Table I.

Figures 1, 2 and 3 compare the advantages of various structures, for $\tau = 1$ and $q = .9$.

Note that in Example A, there is interaction between a_1 and a_2 but no interaction between a_1 and x_2 (or a_2 and x_1). [In Example B, the opposite is true.]

$\alpha_2 \backslash \alpha_1$	(00)	(01)	(10)	(11)
(00)	0 □ □	ρ^2	1 x □	1
(10)	$\rho^2 \tau^2$	$\frac{\rho^2[1+2\rho\tau q+\tau^2]}{1-\rho^2 q^2}$	$\frac{1+2\rho\tau q+\rho^2\tau^2}{1-q^2}$ x ⟶ □	$\frac{1+2\rho\tau q+\rho^2\tau^2}{1-q^2}$
(01)	τ^2 □ x	$\frac{\rho^2+2\rho\tau q+\tau^2}{1-q^2}$ □ ⟵ x	$\frac{1+2\rho\tau q+\tau^2}{1-\rho^2 q^2}$ x x	$\frac{1+2\rho\tau q+\tau^2-q^2(1-\rho^2)}{1-q^2}$ x ⟵ x
(11)	τ^2	$\frac{\rho^2+2\rho\tau q+\tau^2}{1-q^2}$	$\frac{1+2\rho\tau q+\tau^2-q^2\tau^2(1-\rho^2)}{1-q^2}$ x ⟶ x	$\frac{1+2\rho\tau q+\tau^2}{1-q^2}$ x ⟷ x

Table 1. Example A: Best expected gross payoff for all information structures. The corresponding network is indicated for the nine admissible structures.

	(00)	(01)	(10)	(11)
(00)	$a_1 = 0$ $a_2 = 0$	$-\frac{1}{2}\frac{\rho}{\tau}x_2$ 0	$-\frac{1}{2}x_1$ 0	$-\frac{1}{2}x_1$ 0
(10)	$a_1 = 0$ $a_2 = -\frac{1}{2}x_1$	$-\frac{1}{2}\left(\frac{1+q\rho\tau}{1+q^2\rho^2}\right)\frac{\rho}{\tau}x_2$ $-\frac{1}{2}\left(\frac{1+q\frac{\rho}{\tau}}{1-q^2\rho^2}\right)\rho\tau\, x_1$	$-\frac{1}{2}\frac{1+q\rho\tau}{1-q^2}x_1$ $-\frac{1}{2}\frac{\rho\tau+q}{1-q^2}x_1$	$-\frac{1}{2}\left(\frac{1+q\rho\tau}{1-q^2}\right)x_1$ $-\frac{1}{2}\left(\frac{1+q\rho\tau}{1-q^2}\right)x_1$
(01)	$a_1 = 0$ $a_2 = -\frac{1}{2}x_2$	$-\frac{1}{2}\left(\frac{\frac{\rho}{\tau}+q}{1-q^2}\right)x_2$ $-\left(\frac{1}{2}\frac{1+q\frac{\rho}{\tau}}{1-q^2}\right)x_2$	$-\frac{1}{2}\frac{1+q}{1-q^2}\,2\,x_1$ $-\frac{1}{2}\frac{1+2-}{1-q^2\;2}x_2$	$-\frac{1}{2}\left[x_1+\left(\frac{1+q\frac{\rho}{\tau}}{1-q^2}\right)q\,x_2\right]$ $-\frac{1}{2}\left(\frac{1+q\frac{\rho}{\tau}}{1-q^2}\right)x_2$
(11)	$a_1 = 0$ $a_2 = -\frac{1}{2}x_2$	$-\frac{1}{2}\left(\frac{\frac{\rho}{\tau}+q}{1-q^2}\right)x_2$ $-\frac{1}{2}\left(\frac{1+q\frac{\rho}{\tau}}{1-q^2}\right)x_2$	$-\frac{1}{2}\left(\frac{1+q\rho\tau}{1-q^2}\right)x_1$ $-\frac{1}{2}\left[\left(\frac{1+q\rho\tau}{1-q^2}\right)q\,x_1+x_2\right]$	$-\frac{1}{2}\left[\frac{x_1+q\,x_2}{1-q^2}\right]$ $-\frac{1}{2}\left[\frac{qx_1+x_2}{1-q^2}\right]$

Table II. Example A: Best rules of action for all information structures. In each cell $\hat{\alpha}_1$ is at the top, $\hat{\alpha}_2$ is at the bottom.

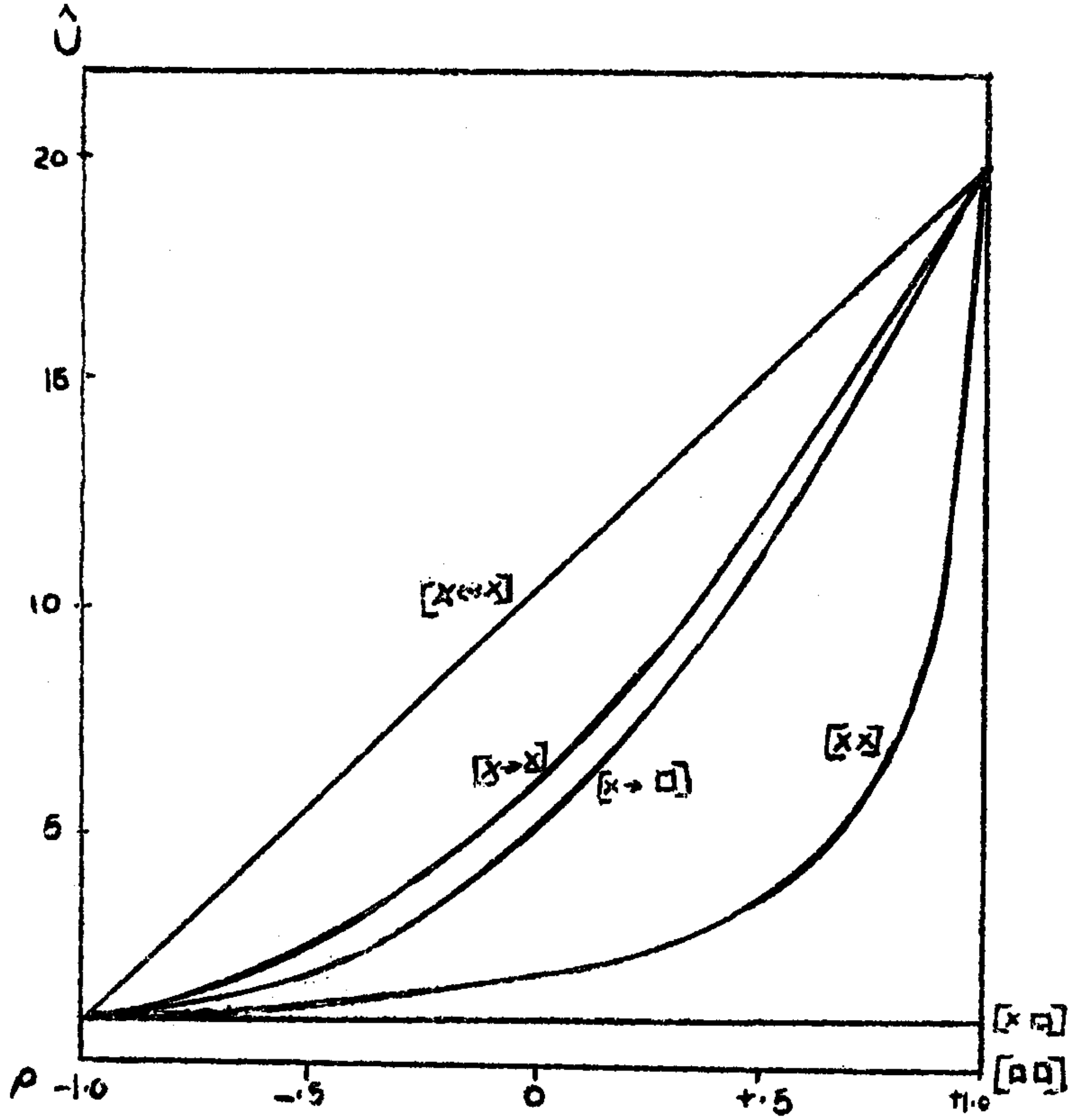

Figure I. Example A: Best gross expected payoffs for various networks.

$\tau = 1, \quad q = .9$

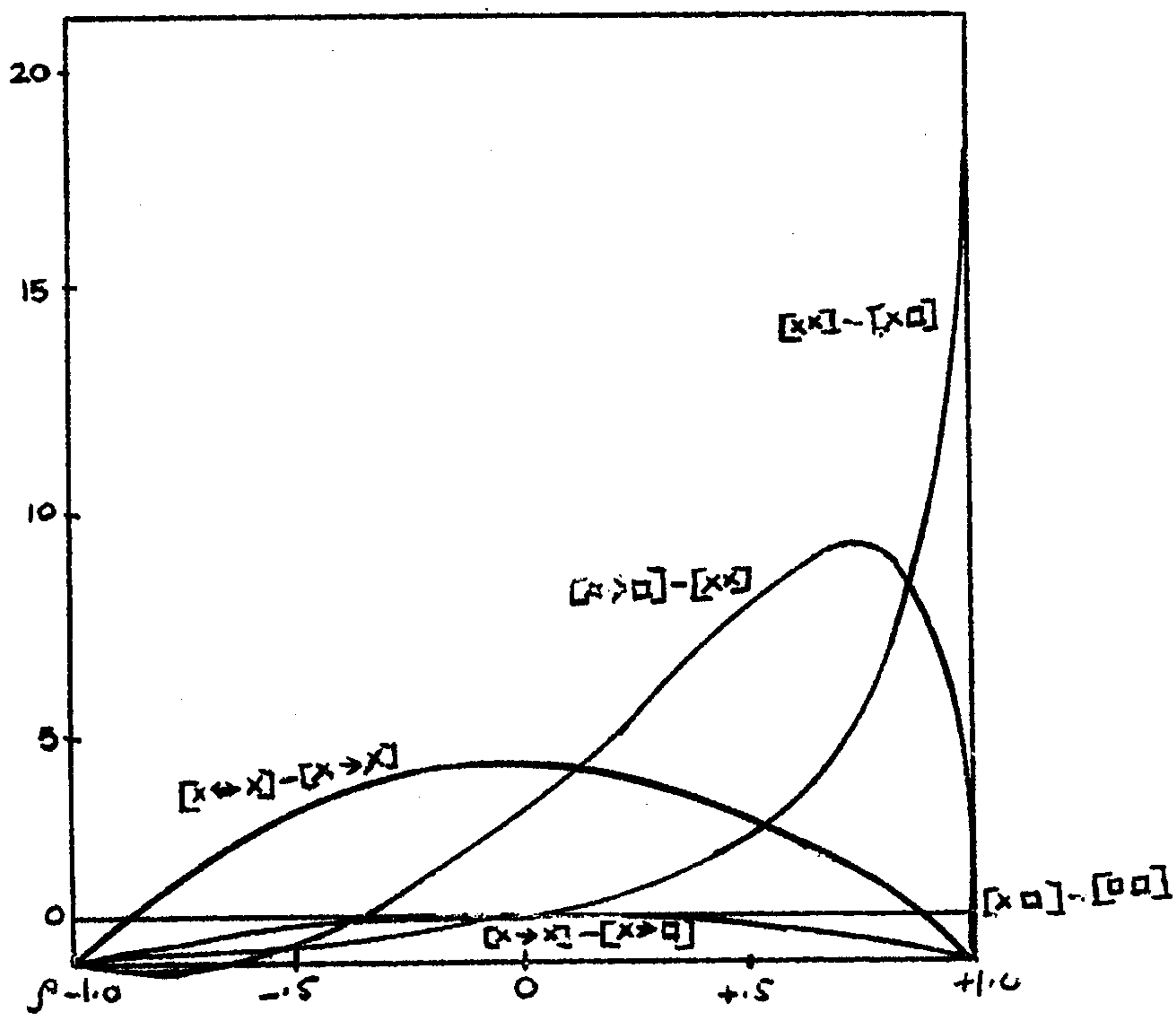

Figure 2. Example A: Successive differences between best gross payoffs for various networks.

$\tau = 1, \quad q = .9$

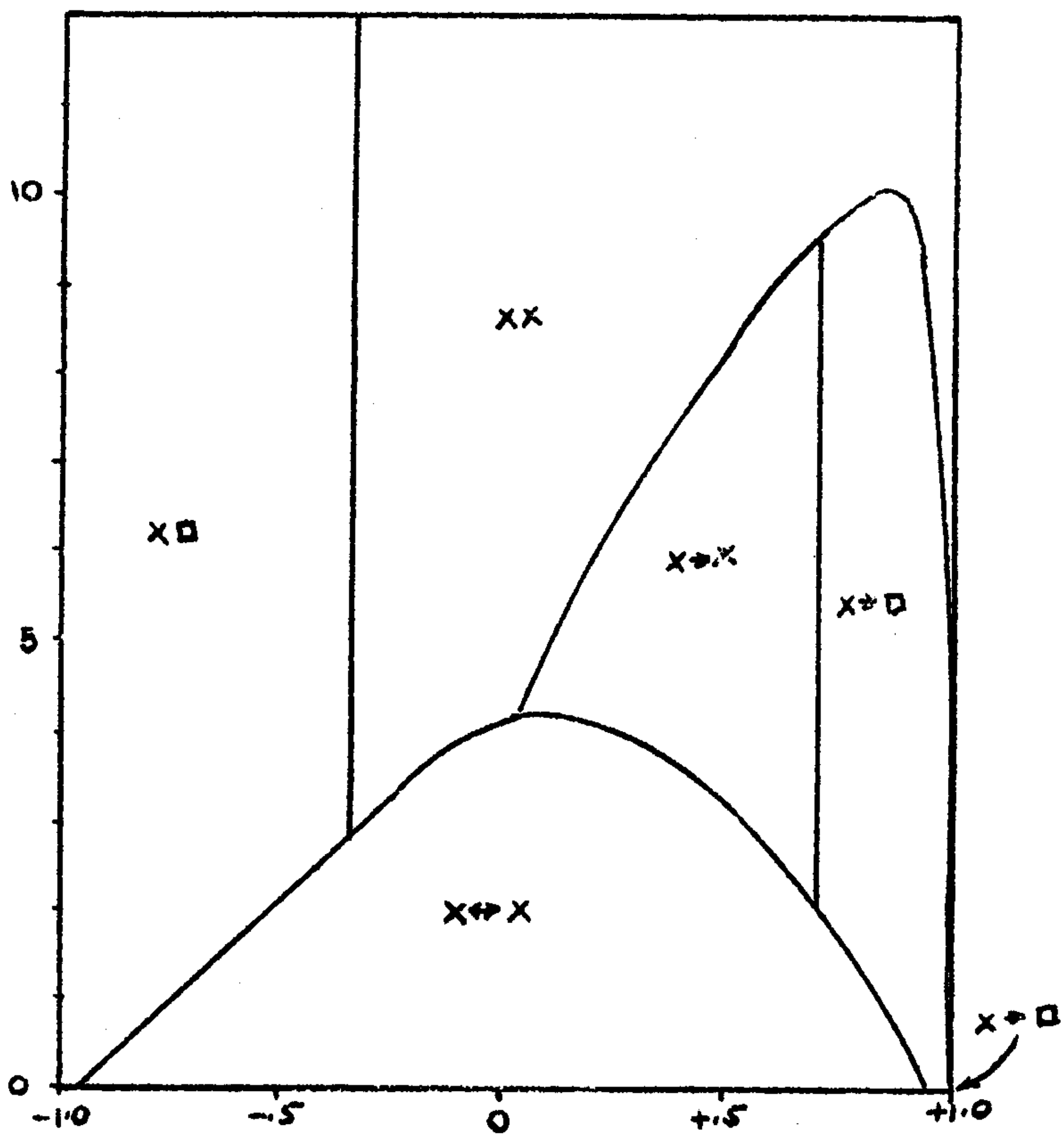

Figure 3. Example A: Regions in ρ - c plane correspond to different optimal networks.

c = cost of one communication link.

cost of observation = 0.5.

$T = 1, \quad q = .9$

III. <u>Example B. Speculative shipping of a commodity.</u>

x_i, predicted price in city i, $i = 1, \ldots, n$

a_{ij}, amount shipped from i to j; $0 \leq a_{ij}$, $\sum_j a_{ij} = 1$ carload

t_{ij}, shipping cost from i to j

$u(a,x) = \sum_i \sum_j a_{ij} (x_j - x_i - t_{ij})$, gross profit

$y^{(i)}$, vector of those predicted prices known to manager in i-th city.

For fixed information structure the best decision rule for the i-th manager is: ship 1 carload to that city j for which $\mathcal{E}(x_j | y^{(i)}) - t_{ij}$ is maximum[2/] this corresponds to condition (1) above].

Best expected gross payoff for a given information structure is

$$\max_{\alpha} U(\alpha, S) = \sum_i \mathcal{E}\left\{\max_j \; \mathcal{E}(x_j - x_i - t_{ij} | y^{(i)})\right\}.$$

The contributions of the different managers are independent and additive (there is no interaction between them). Advantage of "cooperation," if any, must arise from cooperation in collecting and exchanging information.

<u>Special case</u>: n=2

x_i normally distributed with equal means and equal variances, correlation ρ.

t, transportation cost.

The following networks are considered as generating the various information structures: no communication, 1-way communication (either way), 2-way communication. See Figures 4 and 5.

If there are decreasing costs of communication, i.e., 2-way communication costs less than twice as much as 1-way, then there will never be 1-way communication.

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2. It is understood that j may equal i, and that $t_{ii}=0$.

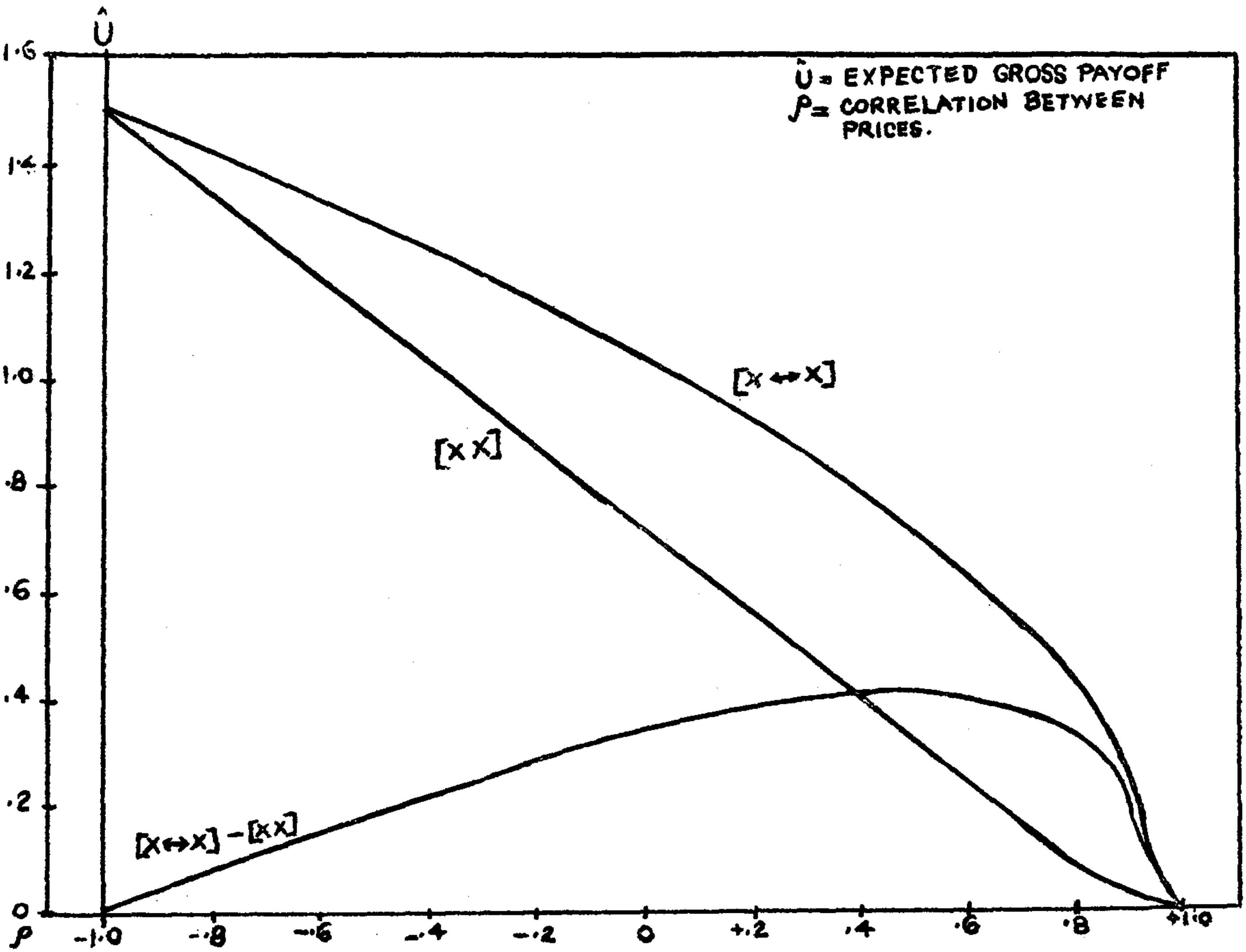

Figure 4. Example B: Speculation: Best expected gross payoffs for various information structures.

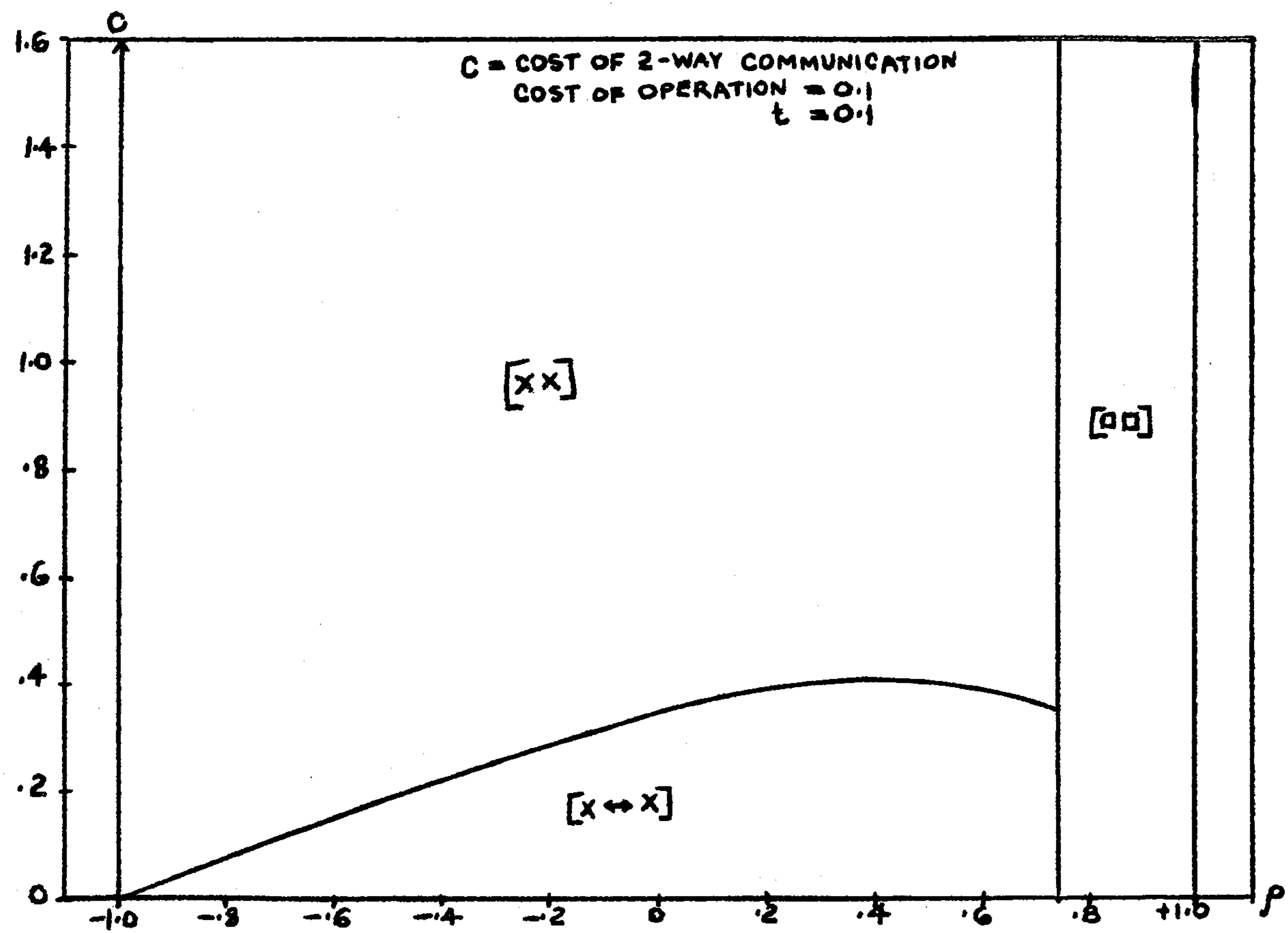

Figure 5. Example B: <u>Speculation</u>: Regions in ρ-c plane corresponding to different optimal networks.

IV. FURTHER DEVELOPMENT.

1. Static Theory: Study of other types of information structure. The "coding" problem [especially: a) the problem of "when not to call" and b) the problem of "what best index-numbers to convey"]. Continuously varying structures.

2. Dynamic Theory: Types of networks and their costs. Iterative procedures for maximisation. Statistical inference carried on concurrently with decision making. Feedback, control of errors.

SUMMARY

Advantages of "teamwork" depend on the payoff function, viz., on the presence of

1. Interaction between man and man ($\partial^2 u / \partial a_i \partial a_j$; $i \neq j$)

2. Interaction between man and nature ($\partial^2 u / \partial a_i \partial x_j$),

together with decreasing costs of cooperation in collecting, processing and exchanging information.

The paper also indicates how these advantages depend on the environment, i.e., on the properties of the probability distribution of external variables. For example, making observations on an external variable is the more worthwhile the larger its variance. The larger the correlation (positive or negative) between two variables the less worthwhile is it to add information about the value of the second variable if the decision-maker knows already the value of the first. These results confirm our intuitive expectations. Yet not all such results can be derived directly from intuition.