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The Generalized (Weak) Le Chatelier Principle in Linear Activity Analysis [1]

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Consider an activity analysis problem

$$a'x = \max_{x}$$

subject to

$$B\,x \stackrel{<}{=} c$$

$$x \stackrel{>}{=} 0.$$

Samuelson [Comparative Statics and the Logic of Economic Maximizing, Rev. Economic Studies, vol. XIV (1), No. 35, 1946-47, p. 42] has shown that for a altered problem with

$$(a + \delta a)'x$$

as the maximand, the solution $x + \quad x$ satisfies an inequality

$$\delta a' \, \delta x \stackrel{>}{=} 0.$$

[1] Research undertaken under contract between the Cowles Commission for Research in Economics and the RAND Corporation.

This is but a special case of the so-called weak Le Chatelier Principle which applies to any constrained maximization problem with linear maximand.

By means of the duality principle it is possible to derive a similar inequality for the change in efficiency prices. However one may ask more generally, what is the effect of simultaneous changes in all parameters (value vector, availability vector, and technology matrix) on the solution and efficiency prices of a given activity analysis problem? The following answer is derived straight from the saddle-point property of the Lagrangean function

$$a'x + \lambda'(c - Bx) = \underset{\lambda}{\mathrm{Min}} = \underset{x}{\mathrm{Max}} .$$

In the following $x + \delta x$, $\lambda + \delta\lambda$ will denote the solution and efficiency prices of the modified problem

$$(a + \delta a)' u = \underset{u}{\mathrm{Max}}$$

subject to

$$(B + \delta B) u \leqq c + \delta c \qquad u \geqq 0 ;$$

Now $a'x + \lambda'[c - Bx] \geqq a'(x + \delta x) + \lambda'[c - B(x + \delta x)]$

$$\text{or} \quad 0 \geqq (a' - \lambda' B)\,\delta x .$$

$$(a + \delta a)' x + (\lambda + \delta\lambda)'[c + \delta c - (B + \delta B)x] \leqq (a + \delta a)'(x + \delta x) + (\lambda + \delta\lambda)'[c + \delta c - (B + \delta B)(x + \delta x)]$$

$$\text{or } 0 \leqq (a' + \delta a')\,\delta x - (\lambda' + \delta\lambda')[B + \delta B]\,\delta x$$

Therefore $-\delta a'\delta x + (\lambda' + \delta\lambda')\,\delta B\,\delta x + \delta\lambda' B\,\delta x \leqq (a' - \lambda' B)\,\delta x \leqq 0$ (1)

Again $a'x + \lambda'[c - Bx] \leqq a'x + (\lambda + \delta\lambda)'[c - Bx]$

$$\text{or} \quad 0 \leqq \delta\lambda'[c - Bx] .$$

$$(a + \delta a)'(x + \delta x) + \lambda'[c + \delta c - (B + \delta B)(x + \delta x)]$$
$$\geqq (a + \delta a)'(x + \delta x) + (\lambda' + \delta \lambda')[c + \delta c - (B + \delta B)(x + \delta x)$$

$$\text{or} \quad 0 \geqq \delta\lambda'[c + \delta c - (B + \delta B)(x + \delta x)]$$

and so $\quad -\delta\lambda'\,\delta c + \delta\lambda'\,B(x + \delta x) + \delta\lambda'\,B\,\delta x \geqq \delta\lambda'[c - Bx] \geqq 0$. (2)

Subtract (1) from (2)

$$\delta a'\,\delta x = \delta\lambda'\,\delta c - \lambda'\,\delta B\,\delta x + \delta\lambda'\,\delta B\,x \geqq 0$$

$$[\delta a' - \lambda'\,\delta B]\,\delta x - \delta\lambda'[\delta c - \delta B\,x] \geqq 0 \qquad (3)$$

Inequality (3) is our generalized weak Le Chatelier Principle.

Let the activity analysis problem be interpreted as that of allocating standing equipment to production processes in a plant:

x	denotes	levels of processes
B	"	uses of fixed equipment per unit levels of processes
c	"	capacities of pieces of equipment
a	"	profitability of processes
λ	"	shadow prices (rentals) on use of equipment

We are considering the effects of changes in profitability, capacity, and the production technology on process levels and shadow rentals.

The term $\delta a' - \lambda'\,\delta B$ denotes the net change in profitability after current rental for equipment has been deducted. The other term $\delta c - \delta B\,x$ denotes the net increase of capacity at current process levels after allowance is made for changes of technology.

(3) Therefore states that either process levels tend to increase with the net profitability or efficiency rents tend to decrease with the net cap city or both. For instance if the only parameter changed is one b_{ij} :

$$\delta b_{i_o j_o} < 0 \quad , \text{ we have}$$

$$- \lambda_{i_o} \quad \delta b_{i_o j_o} \quad \delta x_{jo} + \quad \delta \lambda_{i_o} \quad \delta b_{i_o j_o} \, x_j \geq 0$$

That is, either $\delta x_{jo} \geq 0$ or $\delta \lambda_{io} \leq 0$ or both.

As a result of an improvement in one input coefficient either the process involved is expanded or the shadow rent in question is decreased,or both takes place. This picture remains basically true if several changes in technology occur together. No statement is made about the adjustment of all other process levels and shadow prices. While the Le Chatelier Principle describes reactions one would expect, it should be mentioned that they apply to changes of any (finite) size.

In the case of a quadratic maximand $x'Qx + a'x$ with negative semi-definite Q we obtain by a similar procedure an inequality

$$[(2x + \delta x)' \; \delta Q + \delta a' \;\; - \lambda' \delta B \,] \; \delta x - \delta \lambda' \, [\delta c - \delta Bx \,] \geq 0. \tag{4}$$

Thus if $\delta Q = 0$ our old formula (3) is preserved. Q must be negative semi-definite in order that the Lagrangean function have a maximum at the point where $x'Qx + a'x$ is maximized subject to its constraints. The term

$$(2x + \delta x)' \, \delta Q + \delta a' - \lambda' \, \delta B$$

again designates the net change of profitability.