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Representation of a Preference Ordering by  
a Numerical Function

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1. Introduction

It has often been assumed in economics that if a set  $X$  (Usually in the finite Euclidean space of commodity bundles) is completely ordered by the preferences of some agent, it is always possible to define on that set a real-valued order-preserving function (utility, satisfaction). This is easily seen to be false<sup>2/</sup>.

The particular case where there exists on  $X$  (the set of prospects) a certain algebra of combining (corresponding to the combination of probabilities) has been rigorously and extensively studied by J. von Neumann and O. Morgenstern [7], J. Marschak [6], I. N. Herstein and J. Milnor [5].

But, rather paradoxically, the general case, which is more basic and simpler, has received little attention from economists. H. Wold's study [8] indeed seems to be the only rigorous one; its assumptions are however restrictive.

This note gives conditions under which a complete order (denoted by  $\leq$ ) can be represented by a numerical function. The most common preference ordering in economics is that of bundles of  $n$  commodities, i.e., of points of an  $n$ -dimensional Euclidean space. We shall however treat the problem in a more general frame since this involves no additional mathematical cost.

The familiar case of a set in a finite Euclidean space is covered by the following proposition which is a very special application of theorem II below:

Let  $X$  be a completely ordered subset of a finite Euclidean space. If

for every  $x' \in X$  the sets  $\{x \in X \mid x \leq x'\}$ ,  $\{x \in X \mid x' \leq x\}$  are closed (in  $X$ ), there exists on  $X$  a continuous, real, order-preserving function.

The assumption that the set  $\{x \in X \mid x' \leq x\}$  is closed (in  $X$ ) is equivalent to the more intuitive assumption: let  $(x^k)$  be any sequence of points in  $X$  having a limit  $x^0 \in X$ , if for all  $k$ ,  $x^k$  is at least as good as  $x'$ , then  $x^0$  is at least as good as  $x'$ .

### 2. Two Representation Lemmas

A complete ordering on  $X$  is, to be precise, a binary relation, denoted  $\leq$ , satisfying

- 1) Given any two elements  $x, y$  of  $X$ ;  $x \leq y$  and/or  $y \leq x$
- 2) Given three elements of  $X$  such that  $x \leq y, y \leq z$  then  $x \leq z$ .

From this relation can be derived two new ones:

$x \sim y$  ( $x$  indifferent to  $y$ ) if  $x \leq y$  and  $y \leq x$

$x < y$  ( $y$  better than  $x$ ) if  $x \leq y$  and not  $y \leq x$ .

The quotient set  $x/\sim$ , i.e. the set of indifference classes in  $X$ , will be denoted by  $A$ <sup>3/</sup>. The trivial case where all elements of  $X$  are indifferent (i.e. where  $A$  has just one element) will always be excluded.

The interval  $[x', y']$  is the set  $\{x \in X \mid x' \leq x \leq y'\}$ .

The interval  $]x', y'[$  is the set  $\{x \in X \mid x' < x < y'\}$ .

A real-valued function  $\varphi(x)$  defined on  $X$  is said to be order-preserving if  $x \leq y$  is equivalent to  $\varphi(x) \leq \varphi(y)$ .

A natural topology on  $X$  is a topology<sup>4/</sup> for which the sets  $\{x \in X \mid x \leq x'\}$ ,  $\{x \in X \mid x' \leq x\}$  are closed for all  $x' \in X$ .

Lemma I. Let  $X$  be a completely ordered set whose quotient  $A$  is countable.

There exists on  $X$  a real, order-preserving function, continuous<sup>4/</sup> in any natural topology.

Rank the elements of  $A$ ; it is clearly possible to construct by induction on the rank an order-preserving function  $\psi$  taking  $A$  into some finite real interval. Let  $\lambda = \inf_{a \in A} \psi(a)$ ,  $\mu = \sup_{a \in A} \psi(a)$ .

If  $\alpha'$  satisfies  $\lambda < \alpha' < \mu$  and  $\alpha' \notin \psi(A)$ , four cases may occur: the set  $\{\alpha \in \psi(A) \mid \alpha < \alpha'\}$  (1) may or (2) may not have a largest element, and the set  $\{\alpha \in \psi(A) \mid \alpha' < \alpha\}$  (1') may or (2') may not have a smallest element. We wish to eliminate the gaps type (1-2'), (2-1') and (2-2'); this can easily be done by means of a non-decreasing step function  $\theta(\alpha)$ , the height of each step being equal to the length of the corresponding gap. The new function  $\tilde{\psi}(a) = \psi(a) - \theta[\psi(a)]$  is still order-preserving and  $\tilde{\psi}(A)$  has no gaps of the unwanted types. Denote by  $a(x)$  the indifference class  $a$  to which  $x$  belongs; we finally define  $\varphi(x) = \tilde{\psi}[a(x)]$ . To show that  $\varphi$  is continuous in any natural topology on  $X$  consider a number

$\alpha'$ ,  $\lambda < \alpha' < \mu$  and the set  $X_{\alpha'} = \{x \in X \mid \varphi(x) \leq \alpha'\}$ .

1) If  $\alpha' \in \varphi(X)$ , let  $x' \in X$  be such that  $\alpha' = \varphi(x')$ .

$X_{\alpha'} = \{x \in X \mid x \leq x'\}$  and is therefore closed.

2) If  $\alpha' \notin \varphi(X)$  and if the set  $R_{\alpha'} = \{\alpha \in \varphi(X) \mid \alpha < \alpha'\}$  has a largest element  $\alpha''$ ,  $X_{\alpha'} = X_{\alpha''}$  which is closed by 1).

3) If  $\alpha' \notin \varphi(X)$  and if the set  $R_{\alpha'}$  has no largest element, then the set  $R^{\alpha'} = \{\alpha \in \varphi(X) \mid \alpha' < \alpha\}$  has no smallest element since

$\varphi(X)$  has no gap of type (2 - 1'). Thus  $X_{\alpha'} = \bigcap_{\alpha \in R^{\alpha'}} X_{\alpha}$  and  $X_{\alpha'}$

is closed as an intersection of closed sets.

Similarly one proves that for any number  $\alpha'$  the set  $X^{\alpha'} = \{x \in X \mid \alpha' \leq \varphi(x)\}$  is closed. It follows that the inverse image by  $\varphi$  of any closed set of the real line  $R$  is a closed set of  $X$ .

Lemma II. Let  $X$  be a completely ordered set,  $Z = (z_0, z_1, \dots)$  a countable

subset of X. If for every pair x, y of elements of X such that x < y, there is an element z<sub>1</sub> of Z such that x ≤ z<sub>1</sub> ≤ y, then there exists on X a real, order-preserving function, continuous in any natural topology.

The assumption made is a weakening of the postulate (x < z<sub>1</sub> < y) used by G. Cantor in [3].

Take first the quotient sets X/∼ = A and Z/∼ = C. C is clearly countable and plays for A the role that Z played for X. If A has a smallest and/or a largest element, we can assume, without any loss of generality, that they are contained in C.

Define a new equivalence relation ≍ among elements of A. a ≍ b: between a, b there is a finite number of elements of A. The binary relation ≍ is indeed reflexive, symmetric and transitive. Equivalence classes for ≍ are denoted by a, b, ....

Every equivalence class is clearly countable. Moreover an equivalence class C containing more than one element of A contains an element of C and thus the equivalence classes C form a countable set. Summing up, C' the union of the classes C, is countable and so is D = C ∪ C'.

Construct now on D the function  $\tilde{\varphi}$  as in the proof of Lemma I.  $\tilde{\varphi}$  is extended from D to A as follows. Let a ∈ A and a ∉ D; the set  $D_a = \{d \in D \mid d < a\}$  has no largest element. To see this consider any d' ∈ D<sub>a</sub>. Since a ∉ D, a ∉ C' and there is an infinity of elements of A between d' and a, there is therefore an infinity of elements of C, i.e. of D, between d' and a. Similarly the set  $D^a = \{d \in D \mid a < d\}$  has no smallest element. As a consequence the values  $\sup_{d \in D_a} \tilde{\varphi}(d)$  and  $\inf_{d \in D^a} \tilde{\varphi}(d)$  are not taken on. Moreover these two values are equal since  $\tilde{\varphi}(D)$  has no gap of the (2 - 2') type; they define  $\tilde{\varphi}(a)$ . The function  $\tilde{\varphi}(a)$ , and therefore the function  $\varphi(x) = \tilde{\varphi}[a(x)]$ , are clearly order-preserving and, since  $\varphi(X) = \tilde{\varphi}(A)$  has no gaps of types (1-2') or (2-1'),  $\varphi(x)$  is continuous in any natural topology on X (the proof is the same as for Lemma I).

### 3. Two Representation Theorems

Before stating Theorem I we recall two definitions. A (topological) space  $X$  is separable if it contains a countable subset whose closure is  $X$ . A (topological) space  $X$  is connected if there is no partition of  $X$  into two disjoint, non-empty, closed sets.

Theorem I Let  $X$  be a completely ordered, separable, and connected space.  
If for every  $x' \in X$  the sets  $\{x \in X \mid x \leq x'\}$  and  $\{x \in X \mid x' \leq x\}$  are  
closed, there exists on  $X$  a continuous, real, order-preserving function.<sup>5/</sup>

This theorem can easily be derived from the results of S. Eilenberg [4]. It will be proved here as an immediate consequence of Lemma II. A much more direct proof could assuredly be given: the motivation for the two lemmas is Theorem II.

Call  $Z$  the countable set dense in  $X$  and consider a pair  $x', y'$  of elements of  $X$  such that  $x' < y'$ . The sets  $\{x \in X \mid x \leq x'\}$  and  $\{x \in X \mid y' \leq x\}$  are disjoint, non-empty and closed, they cannot exhaust  $X$  which is connected, therefore the open interval  $]x', y'[$  is not empty, and it must contain an element  $z_1 \in Z$ . The theorem is proved since the topology on  $X$  is a natural topology.

The assumption of connectedness is however very strong. We give a second theorem where it is removed at the cost of a slightly stronger separability assumption.

A topological space  $X$  is perfectly separable if there is a countable class  $(S)$  of open sets such that every open set in  $X$  is the union of sets of the class  $(S)$ .

We remark that a separable metric space is perfectly separable, that a subspace of a perfectly separable space is perfectly separable.

Theorem II Let  $X$  be a completely ordered, perfectly separable space.

If for every  $x' \in X$  the sets  $\{x \in X \mid x \leq x'\}$  and  $\{x \in X \mid x' \leq x\}$  are closed, there exists on  $X$  a continuous, real, order-preserving function.

Choose an element in each non-empty set  $S$ ; they form a countable set  $Z''$ .

Consider then the pairs  $a', b'$  of elements of  $A$  such that  $a' < b'$  and the interval  $]a', b'[$  is empty. The set of those pairs is countable. To see this, associate with each such pair a set  $S_{b'}$ , as follows: take two elements  $x', y'$  in the indifference classes  $a', b'$  respectively. The set  $\{x \in X \mid x < y'\}$  is open and therefore there exists a set  $S_{b'}$  in the class (S) such that  $x' \in S_{b'}$ ,  $C \{x \in X \mid x < y'\}$ . If  $a'', b''$  is another pair with the same properties,  $S_{b''}$  is different from  $S_{b'}$ , for one has  $a' < b' \leq a'' < b''$ , in which case  $x'' \in S_{b''}$  and  $x'' \notin S_{b'}$ , or  $a'' < b'' \leq a' < b'$ , in which case  $x' \in S_{b'}$  and  $x' \notin S_{b''}$ . The pairs  $a', b'$  are thus in one-to-one correspondence with a subclass of the countable class (S). Choose then an element  $x'$  in each class  $a'$  and an element  $y'$  in each class  $b'$ . All those  $x'$  and  $y'$  form a countable set  $Z'$ .

Consider finally the countable set  $Z = Z' \cup Z''$ ; it has all the properties required by Lemma II. Let  $x, y$  be a pair of elements of  $X$  such that  $x < y$ . If the open set  $]x, y[$  is not empty, it contains a non-empty set  $S$  and therefore an element of  $Z''$ . If the set  $]x, y[$  is empty,  $x \sim x' \in Z'$  and  $y \sim y' \in Z'$ . So that in any case  $]x, y[$  contains an element of  $Z$ .

Footnotes

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2. Consider the lexicographic ordering of the plane: a point of coordinates  $(a', b')$  is better than the point  $(a, b)$  if " $a' > a$ " or if " $a' = a$  and  $b' > b$ ". Suppose that there exists a real order-preserving function  $\alpha(a, b)$ . Take two fixed numbers  $b_1 < b_2$  and with a number  $a$  associate the two numbers  $\alpha_1(a) = \alpha(a, b_1)$  and  $\alpha_2(a) = \alpha(a, b_2)$ . To two different numbers  $a, a'$  correspond two disjoint intervals  $[\alpha_1(a), \alpha_2(a)]$  and  $[\alpha_1(a'), \alpha_2(a')]$ . One obtains therefore a one-to-one correspondence between the set of real numbers (non-countable) and a set of non-degenerate disjoint intervals (countable).
3. For definitions relating to an equivalence relation see [1].
4. For definitions of a topology and of a continuous function see [2, § 1] and [2, § 4] respectively.
5. The closedness assumptions have already been used in a similar context by I. N. Herstein in an earlier unpublished version of [5].



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