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The Problem of Constructing Indicators of Goodness

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Before a subject can make good choices between different plans $(i = 1,2,\dots)$ to continue a sequence of moves $a_{\alpha,t\rfloor}$ already made he must make an evaluation of his estimates $P_t^{(h;i)}$ of the probability of different subsets of possible past and future developments under the condition that he makes the decision to try to realize the plan (i).

He needs then an indicator of goodness g_i with the property $g_i > g_j$, whenever the plan i *is* better than j.

It does not matter if he instead of the indicator g_i uses some other indicator $U(g_i)$ with the property $U(g_i) > U(g_j)$ if $g_i > g_j$. The scaling and origin of an indicator of goodness is thus at our free disposal when we try to construct indicators of goodness. Indicators of goodness has been called utility functions, preference scales, etc. We will, however, use the term preference scale as a general term and try to construct indicators of goodness, where the origin $g_i = 0$ and the scaling is such that the numerical value of the indicator has some interesting interpretation.

It seems to be true that nature has given every decision maker some crude preference scale as a gift to enable him to make purposive decisions and thereby increase the probability of survival for the decision maker and the set of human beings to which he is related by parenthood, kinship, friendship and other ties. But this preference scale is only indirectly observable by studying the situations and acts really made by others and directly by changes in the feelings of the decision-maker when he anticipates the consequences of different possible decisions. If he could construct a reliable indicator of goodness as a mathematical function of his information about the past and the different possible plans of action he would be able to make better decisions, than without such an instrument.

The task of constructing such an indicator of goodness is by no means easy. Yet it is quite worth while to search for indicators of goodness which have a higher serial correlation with an ideal indicator of goodness than the crude indicators we already have as a gift of nature modified to some degree by the social pressure field in which we have lived. Even in the case when no practical use can be made of a mathematically formulated indicator of goodness, because the calculations necessary would be too complicated or too slow for the decision making in question, such a formulation might still be useful in improving those "short cuts" actually used in the judgement phase of decision making.

The ideal indicator of goodness g_i we thus postulate as a function

$$g_i = g(x_{\alpha \underline{t|}}, a_{\alpha \underline{t|}}; a_{i \underrightarrow{|t}})$$

of the information $x_{\alpha \underline{t|}}$ the subject A_α has about the past, his own actions $a_{\alpha \underline{t|}}$ in the past, and the plan $a_{i \underrightarrow{|t}}$ he is trying to evaluate.

From his information about the past we assume that the decision maker is able to estimate by means of some information transforming process a probability distribution

$$P^{(h,i)}_{\alpha t} \approx P(x_{\alpha \lfloor t} \subset H_h;\ a_{i \lfloor t \rightarrow};\ x_{\alpha\ t \rfloor},\ a_{\alpha\ t \rfloor})$$

for the consequences $x_{\alpha \lfloor t}$ of the decision to try to realize the plan $a_{i \lfloor t \rightarrow}$.

The problem of constructing the function g_i can now be reformulated as a problem of evaluating the goodness of a set of estimated conditional probability distributions $P^{(\cdot;i)}_{\alpha t}$.

$$g_i = g(P^{(\cdot;i)}_{\alpha t}).$$

1. First step in the construction of an indicator of goodness.

We start the construction of g_i by considering only two possible decisions $i = 0$ and $i = 1$, to which correspond two degenerated probability distributions $P^{(o)}_{\alpha t}$ and $P^{(1)}_{\alpha t}$. By a degenerated probability distribution over the space of all events we understand the distribution which gives the measure 1 to a uniquely defined course of events and the measure 0 to all other courses of events. This is the assumed case of complete information.

To $P^{(o)}_{\alpha t}$ corresponds then the course of events x_o and to $P^{(1)}_{\alpha t}$ the course of events x_1 assumed known for the whole past and future. In this case

$$g_o = g(x_o),\ g_1 = g(x_1).$$

We have however to remember that x_o and x_1 are not simple numbers but denote paths in the space of events. A unique description of a path in an abstract space is possible only by means of a reference system of paths.

We use the path x_o itself as an reference path, and define $g_o = g(x_o) = 0$. The problem reduces itself then to a problem of defining g_1 as a function of x_1 described with reference to x_o. Because we have yet at our free disposal all real numbers except $g_o = 0$, and have not defined the scale for the function $g(x_1)$ we see that we are free to define for instance that $g(x_1) = 1$ if x_1 is <u>better</u> than x_o and $g(x_1) = -1$ if x_1 is inferior to x_o. Before the scale of the function $g(x)$ is fully defined we need a set of paths x_g defined for every g, that can be used as reference paths when we try to define $g(x)$ for paths not belonging to the reference system of paths. A convenient system of reference paths seems to me to be the paths in the space of all courses of events, which can be interpreted as money gains of the amount $g \gtreqless 0$, received by A_α during the short period $(t \longrightarrow t + dt)$ preceded by the path $x^{(\alpha)}_{\underline{t|}}$ until t and followed after $t + dt$ by a path $x^{(\alpha)}_{g\underrightarrow{|t + dt}}$ the two differing from each other only with respect to variables connected with the events of saving or spending the gain g. To these paths x_g we assign the value g. For every A_α and $\underline{t}$ we thus define

$$g = g(x_g) = g(P^{(g)}_{\alpha t})$$

for this specific set of reference paths. By this convention we have only defined a scale for indicators of goodness, that has a conceivable interpretation in the subset $\left\{ P^{(g)}_{\alpha t} \right\}$ of the space of all probability distributions over the space of courses of events. The problem of constructing $g(P^{(i)}_{\alpha t})$ for more general distributions $(P^{(i)}_{\alpha t})$ is thereby not solved.

In a second step we will consider a probability distribution $P(.)$

over the set of paths $\left\{x_g\right\}$, $P(\gamma) = P(g \geq \gamma)$ and try to define the goodness $g(P(\circ))$ of such a distribution $P(\circ)$. The task cannot be solved satisfactorily by simply putting $g(P(\circ)) = Eg$. This quantity:

$$Eg = \int_{g=\infty}^{-\infty} g\,dP(g)$$

is not equal to $g(P(\circ))$ because such an indicator of goodness would give no explanation of obviously good decisions made in connection with insurance, and would give no justification for the purchase of lottery tickets. The gain $g(P(\circ))$, equivalent to the prospect of getting different gains g with a probability $= P(\gamma)$, is clearly a number in the interval (γ_1, γ_0), where $\gamma_1 < \gamma_0$ is the largest number for which $P(\gamma) = 1$ and γ_0 is the smallest number for which $P(\gamma) = 0$. The number $g(P(\circ))$ can therefore be considered as an average of g, but not necessarily an arithmetic average $= Eg$. Such more general averages can be calculated by means of a mirror function $U_t^{(\alpha)}(g)$, as

$$U_t^{(\alpha)}(g_t^{(\alpha)}(P(\circ))) = E\, U_t^{(\alpha)}(g) = \int_{+\infty}^{-\infty} U_t^{(\alpha)}(g)\, dP(g).$$

The mirror function $U_t^{(\alpha)}(g_t^{(\alpha)}(P(\circ))$ can itself be considered as an indicator of goodness (a utility function). The mirror function $U(g)$ is a monotonically increasing function of g, chosen in such a way that the averages of g calculated according to this formula will give indicators of goodness $g_{\alpha(t)}(P(\circ))$ expressed as gains of money equivalent to the prospect of getting gains $g > \gamma$ with a probability $= P(\gamma)$. If we use instead of a specific mirror function $U_t^{(\alpha)}(g)$ a linear transform

$$L(U_t^{(\alpha)}(g) = \chi_1\, U_t^{(\alpha)}(g) + \chi_0$$

this mirror function defines the same average $g_t^{(\alpha)}(P(.))$ as $U_t^{(\alpha)}(g)$ because

$$L(U_t^{(\alpha)}(g_t^{(\alpha)}(P(.)) = E(LU_t^{(\alpha)}(g)).$$

If we like, we can standardize the mirror function by putting $U_t^{(\alpha)}(0) = 0$ and

$$u_t^{(\alpha)}(0) = \int_g^0 \left(\frac{\partial U_t^{(\alpha)}(g)}{\partial \; g} = u_t^{(\alpha)}(g)\right) = 1$$

and call this function $U_t^{(\alpha)}(g)$ the utility function. The partial derivative $u_t^{(\alpha)}(g)$ of this mirror function we could call the sensibility function of A_α at t.

By partial integration of the equation

$$U_t^{(\alpha)}(g_t^{(\alpha)}(P(.))) = \int_{+\infty}^{-\infty} U_t^{(\alpha)}(g)\;dP(g) = \int_{\infty}^{0} U_t^{(\alpha)}(g)\;dP(g) + \int_0^{-\infty} U_t^{(\alpha)}(g)dP(g)$$

we get

$$U_t^{(\alpha)}(g_t^{(\alpha)}(P(.))) = \Big|_{\infty}^{-\infty} U_t^{(\alpha)}(g)\;P\;(g) + \int_{-\infty}^{\infty} P(g)\;u_t^{(\alpha)}(g)\;dg$$

$$= \int_{\gamma>0} P(g > \gamma)\;u_t^{(\alpha)}(\gamma)d\gamma - \int_{\gamma<0} P(g < \gamma)u_t^{(\alpha)}(\gamma)\;d\gamma.$$

The first integral:

$$\int_{\gamma>0} P(g > \gamma)\;u_t^{(\alpha)}(\gamma)\;d\gamma = P(g > 0).\;U_t^{(\alpha)}(g_t^{(\alpha)}(P(.\;|>0))$$

measures the utility value of expected positive gains. This value divided by the probability $P(g > 0)$ of getting positive gains could be considered as the utility value of the gains in the case when no losses $g < 0$ occur.

The second integral

$$- \int_{\gamma<0} P(g < \gamma) u_t^{(\alpha)}(\gamma)\, d\gamma = -P(g < 0)\, U_t^{(\alpha)}(g_t^{(\alpha)}(P(\cdot | < 0)))$$

measures the utility value of expected losses $g < 0$. If we modify the probability distribution $P(\cdot)$ in such a way that A_χ is of the opinion that $g_t^{(\alpha)}(P(\cdot)) \approx 0$ we get a method of measuring $U_t^{(\alpha)}(g_t^{(\alpha)}(P(\cdot | > 0)))$ in relation to $U_t^{(\alpha)}(g_t^{(\alpha)}(P(\cdot | < 0)))$ by means of the formula:

$$\frac{U_t^{(\alpha)}(g_t^{(\alpha)}(P(\cdot | > 0)))}{U_t^{(\alpha)}(g_t^{(\alpha)}(P(\cdot | < 0)))} = -\frac{P(g < 0)}{P(g > 0)} = 1 - \frac{1}{P(g > 0)} .$$

By such measurements we can start with degenerated distributions.

$$P(g > \gamma | > 0) = \begin{cases} 0 & ; \ 0 < g < \gamma \\ 1 & ; \ g > \gamma \end{cases} \text{ and } P(g < \gamma | < 0) = \begin{cases} 0 & \gamma < -1 \\ 1 & 0 > \gamma > -1 \end{cases}$$

in which case $g_t^{(\alpha)}(P(\cdot | > 0) = \gamma$ and $g_t^{(\alpha)}(P(\cdot | < 0) = -1$. We have in this case if $P(g > 0) = P(g = \gamma)$ is chosen in such a way that $g(P(\cdot)) = 0$

$$\frac{U_t^{(\alpha)}(\gamma)}{U_t^{(\alpha)}(-1)} = 1 - \frac{1}{P(g = \gamma)} .$$

There-after it may be possible to study whether for more complicated distributions, for instance a discrete distribution over the number system (γ_1, γ_2); $0 < \gamma_1 < \gamma_2$), $U_t^{(\alpha)}(g_t^{(\alpha)}(P(\cdot | 0)))$ can be approximately calculated as a mean value $(p_{\gamma_1} \cdot U_t^{(\alpha)}(\gamma_1) + p_{\gamma_2} U_t^{(\alpha)}(\gamma_2)) : (p_1 + p_2)$ and so on.

We shall suppose that the results of such studies about the utility values assigned by A_α at t to different probability distributions $P(g)$ can with sufficient accuracy be stated according to the following postulate:

Postulate I. The indicator of goodness $g_t^{(\alpha)}(P(\cdot))$ can be calculated by means of a formula

$$(1) \qquad U_t^{(\alpha)}(g(P(\cdot)) = EU_t^{(\alpha)}(g) = \int_{\gamma>0} P(g>\gamma)\, u_t^{(\alpha)}(\gamma)\, d\gamma - \int_{\gamma<0} P(g<\gamma)\, u_t^{(\alpha)}(\gamma)\, d\gamma$$

where the sensibility

$$(2) \qquad u_t^{(\alpha)}(g) = \frac{\partial U_t^{(\alpha)}(g)}{\partial g} > 0$$

is continuous for every g and independent of the form of the probability distribution $P(\cdot)$.

The mirror function (utility function) $U_t^{(\alpha)}(g)$ is standardized by means of the conventions:

$$(3) \qquad U_t^{(\alpha)}(0) = 0$$

$$(4) \qquad u_t^{(\alpha)}(0) = 1$$

The form of the function $u_t^{(\alpha)}(g)$ and $U_t^{(\alpha)}(g)$ depends on the personality and past experiences of A_α but we have always

$$(5) \qquad U_t^{(\alpha)}(g) + U_t^{(\alpha)}(-g) < 0.$$

The attempts made to study $U_t^{(\alpha)}(g)$ seems further to have confirmed the following information:

$$(6) \qquad \frac{\partial u_t^{(\alpha)}(g)}{\partial g} = 0$$

for two or more negative values of g and one positive g.

(7) $$\frac{\partial u_t^{(\alpha)}(g)}{\partial g} = 0, \quad \frac{\partial^2 u_t^{(\alpha)}(g)}{\partial^2 g} > 0 \quad \text{for } g = 0.$$

The sensibility has a relative minimum for $g = 0$ $(u_t^{(\alpha)}(0) = 1)$.

(8) $$\left|U_t^{(\alpha)}(g)\right| < \infty, \; 0 < u_t^{(\alpha)}(g) < \infty \text{ for every } g.$$

The sensibility function can thus be considered as proportional to a "probability intensity" with many modes, and $U_t^{(\alpha)}(g)$ as a linear function of a cumulative distribution function. This distribution function $F_t^{(\alpha)}(g)$ can in general be considered as a mean value (a mixture) of a set of unimodal (approximately normal) distributions.

(9) $$\frac{U_t^{(\alpha)}(g) - U_t^{(\alpha)}(-\infty)}{U_t^{(\alpha)}(\infty) - U_t^{(\alpha)}(-\infty)} = F_t^{(\alpha)}(g) \approx \sum_i p_{it}^{(\alpha)} \Phi\left(\frac{g - g_{it}^{(\alpha)}}{\sigma_{it}^{(\alpha)}}\right) \; ; \; \Sigma p_{it}^{(\alpha)} = 1$$

The distribution $F_t^{(\alpha)}(g)$ is a mirror function equivalent to the utility function $U_t^{(\alpha)}(g)$.

The quantity:

$$F_t^{(\alpha)}(g_t^{(\alpha)}(P(.))) = \int_{y=-\infty}^{\infty} F_t^{(\alpha)}(y)\, dP(g \leq y).$$

can be considered as meaning the probability that the stochastic variable $z' = y' - x' \lesseqgtr 0$, if x' and y' are two independent stochastic variables with the marginal distribution $P(x' < x) = P(g < x)$ and $P(y' < y) = F_t^{(\alpha)}(y)$.

The probability distribution of z' is

$$P(z' \leq Z) = \int_{x=-\infty}^{\infty} F_t^{(\alpha)}(Z + x)\, dP(g \leq x).$$

As a special case, we get $F_t^{(\alpha)}(g_t^{(\alpha)}(P(.))) = P(z' \leq 0)$. This fact gives a new interpretation to the indicator of goodness $g_t^{(\alpha)}(P(.))$ as that value x of a distribution function $F_t^{(\alpha)}(x)$ which determines the same probability as the half plane $y' \lesssim x'$ in the two dimensional distribution of two independent stochastic variables with the marginal distributions $F_t^{(\alpha)}(.)$ and $P(.)$. This interpretation can perhaps suggest some method for calculation of indicators of goodness when the distributions $F_t^{(\alpha)}(.)$ and $P(.)$ are assumed to be known.

As an example let us assume that

$$F_t^{(\alpha)}(x) = \frac{1}{2}\left(\Phi(x+1) + \Phi\left(\frac{x-100}{10}\right)\right),\quad P(.) = \Phi\left(\frac{x-m}{\sigma}\right)$$

By a straight forward integration we get

$$F_t^{(\alpha)}(g_t^{(\alpha)}(P(.))) = \frac{1}{2}\left(\Phi\left(\frac{1+m}{\sqrt{\sigma^2+1}}\right) + \Phi\left(\frac{m-100}{\sqrt{100+\sigma^2}}\right) \right)$$

$$= \frac{1}{2}\left(\Phi(g_t^{(\alpha)}(.) + 1) + \Phi\left(\frac{g_t^{(\alpha)}(.) - 100}{10}\right) \right).$$

The solution $g_t^{(\alpha)}(.)$ to this equation is approximately (for small m and σ)

$$g_t^{(\alpha)}(.) \lessapprox \frac{1+m}{\sqrt{1+\sigma^2}} - 1 < m,\quad g_t^{(\alpha)}(.) \approx 0 \text{ for } m = \frac{1}{2}\sigma^2.$$

The value $g \approx -1$ corresponds to the highest sensibility for losses that is approximately the wealth of A_α. More generally if:

$$F_t^{(\alpha)}(x) = \Sigma\, c_i\, \Phi\left(\frac{x - m_{it}^{(\alpha)}}{\sigma_{it}^{(\alpha)}}\right)$$

and

$$P(\alpha) = \sum_v p_v\, \Phi\left(\frac{x-m_v}{\sigma_v}\right).$$

we get

$$F_t^{(\alpha)}(g_t^{(\alpha)}(.)) = \sum_i \sum_v c_i p_v \Phi\left(\frac{m_v - m_{it}^{(\alpha)}}{\sqrt{\sigma_v^2 + (\sigma_{it}^{(\alpha)})^2}}\right)$$

$$= \sum c_i \Phi\left(\frac{g_t^{(\alpha)}(P(.) - m_{it}^{(\alpha)}}{\sigma_{it}^{(\alpha)}}\right) .$$

For practical purposes

$$F_t^{(\alpha)}(g_t^{(\alpha)}(.)) = \sum_{i,v} c_i p_v \ \Phi\left(\frac{m_v - m_{it}^{(\alpha)}}{\sigma_{vi}^{(\alpha)}}\right) \ ; \quad \sigma_{vi}^{(\alpha)} = \sqrt{\sigma_v^2 + (\sigma_{it}^{(\alpha)})^2}$$

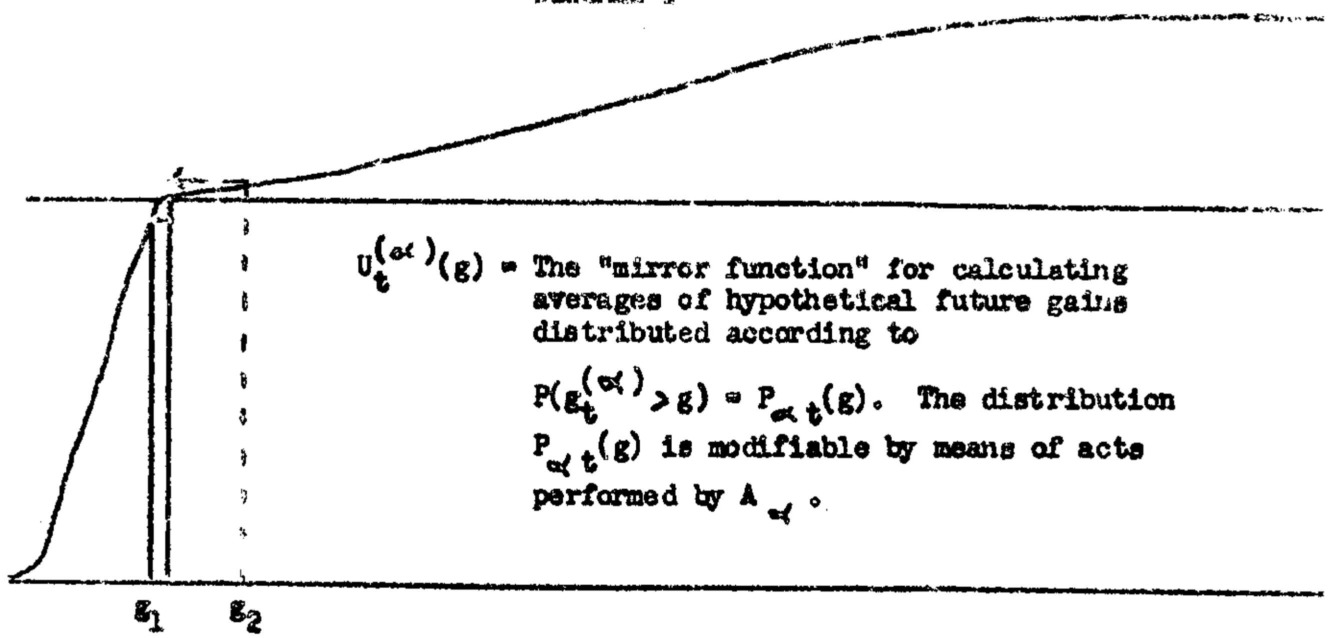
can be used as an indicator of goodness instead of $g_t^{(\alpha)}(.)$ for evaluation of probability distributions (x), which are mixtures of normal distributions with mean values m_v and standard deviations σ_v.

In diagram (1) and (2) we have described some plausible forms for $F_t^{(\alpha)}(x)$ and discussed the economic meaning of the components in the mixture of distributions functions of which $F_t^{(\alpha)}(x)$ is a sum.

Second step in construction of an indicator of goodness g_i. Future events.

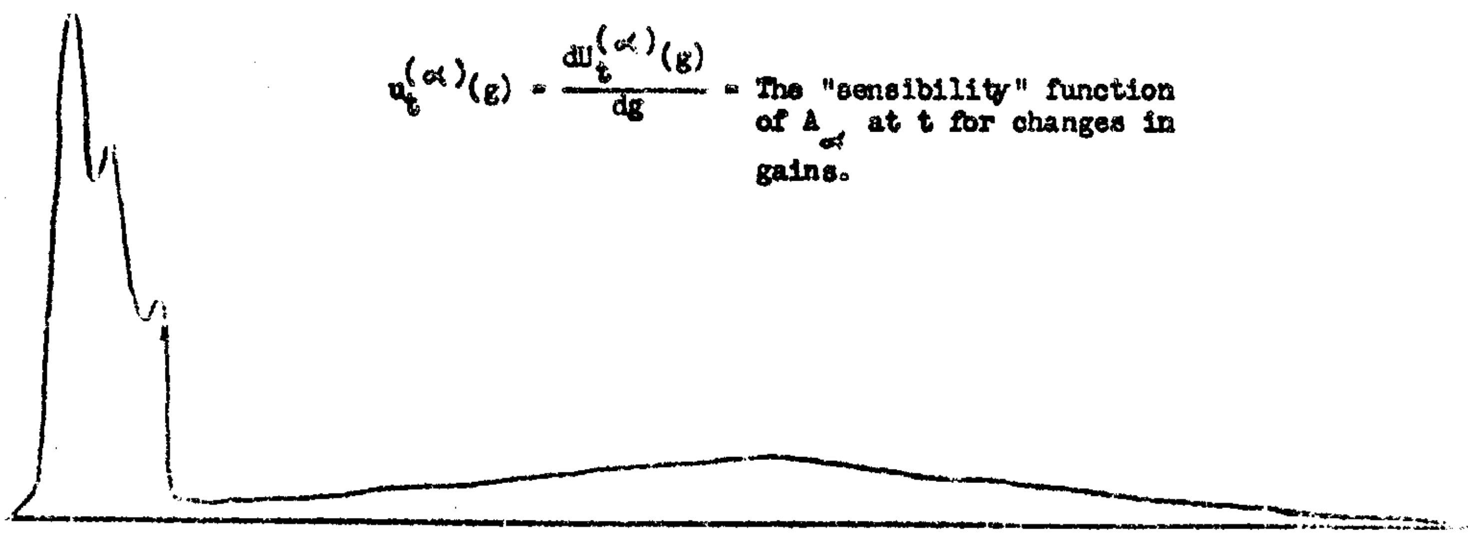
Heretofore we have considered only probability distributions of anticipated money gains g receivable by A_α in the immediate future $t \to t + \Delta t$, where Δt is so short that no "problem of discounting" arise. The second step is to introduce a probability distribution for the point of time $t + \tau_g$ the gain g is anticipated to be received by A_α . The distribution of the amount g itself is degenerated to $P(g) = 1$. If also the distribution of τ_g is degenerated and the probability that A_α is alive at $t + t_g$ is assumed to be = 1, we have the case of simple discounting g at $t + \tau_g$ to gain $g_{\alpha t}(\tau_g) = v_{\alpha t}(\tau_g)$. g received at $t + \Delta t$, increasing the indicator of goodness with the same amount as (g at $t + \tau_g$).

DIAGRAM I

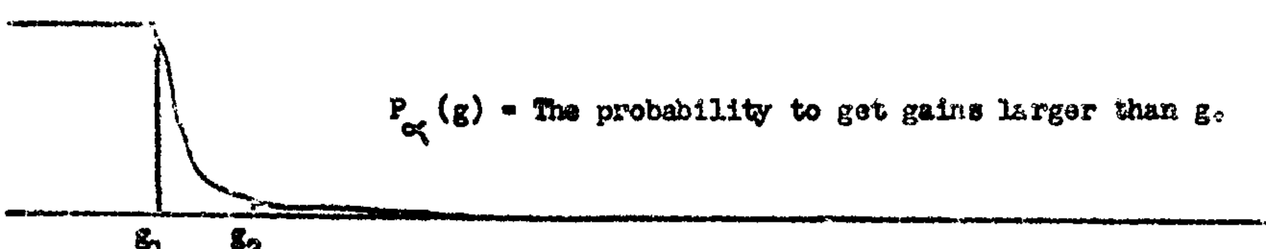


$U_t^{(\alpha)}(g)$ = The "mirror function" for calculating averages of hypothetical future gains distributed according to

$P(g_t^{(\alpha)} > g) = P_{\alpha t}(g)$. The distribution $P_{\alpha t}(g)$ is modifiable by means of acts performed by A_α.



$u_t^{(\alpha)}(g) = \dfrac{dU_t^{(\alpha)}(g)}{dg}$ = The "sensibility" function of A_α at t for changes in gains.



$P_\alpha(g)$ = The probability to get gains larger than g.

$$U_t^{(\alpha)}(g_{tP_{\alpha t}}^{(\alpha)}) = E\,U_t^{(\alpha)}(g_t^{(\alpha)}) = \int_{+\infty}^{-\infty} U_t^{(\alpha)}(g)\,d\,P_{\alpha t}(g) = \int_{-\infty}^{\infty} P_{\alpha t}(g)\,u_t^{(\alpha)}(g)\,dg.$$

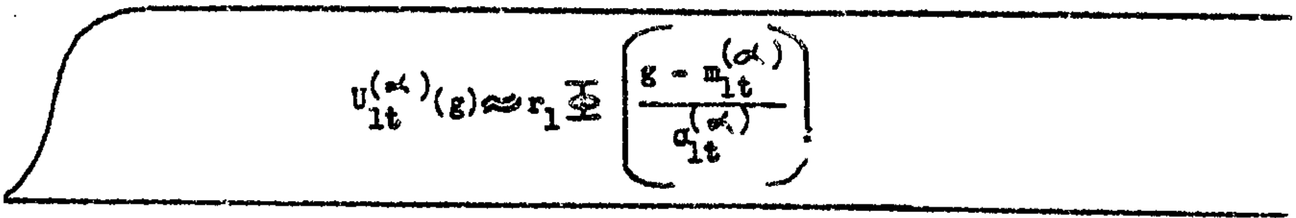
$g_{tP_{\alpha t}}^{(\alpha)}$ = Average gain mirrored through $U_t^{(\alpha)}(g)$.

For the above pictured distribution $P_\alpha(g)$ and $U_t^{(\alpha)}(g)$, $g_{tP_{\alpha t}}^{(\alpha)} \approx 0$

in spite of the fact that the arithmetical mean is clearly > 0.

DIAGRAM II

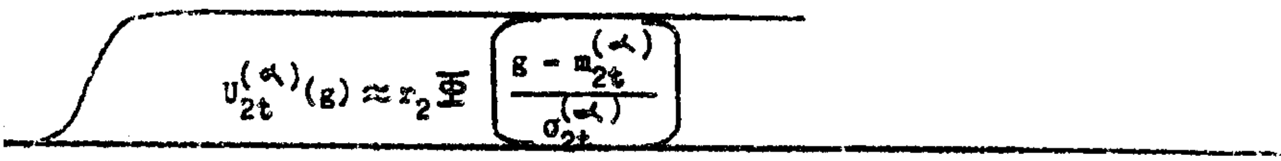
<u>Decomposition of</u> $U_t^{(\alpha)}(g) \approx \sum_{\lambda=1}^{4} U_{\lambda t}^{(\alpha)}(g) = \sum_{\lambda=1}^{4} r_\lambda \Phi\left(\frac{g - m_{\lambda t}^{(\alpha)}}{\sigma_{\lambda t}^{(\alpha)}}\right)$



$$U_{1t}^{(\alpha)}(g) \approx r_1 \Phi\left(\frac{g - m_{1t}^{(\alpha)}}{\sigma_{1t}^{(\alpha)}}\right)$$

$g=0$

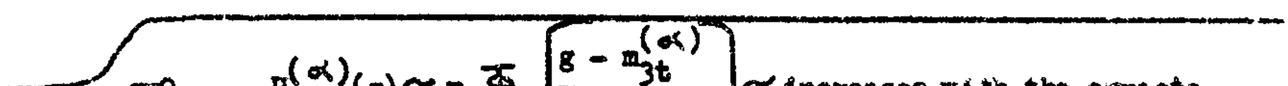
$U_{1t}^{(\alpha)}(g)$ increases with the expectation of A_α <u>not to be</u> "broken down" if he gets a gain = g and lives very carefully and decreases with the decrease in well-being necessary if $g < 0$.



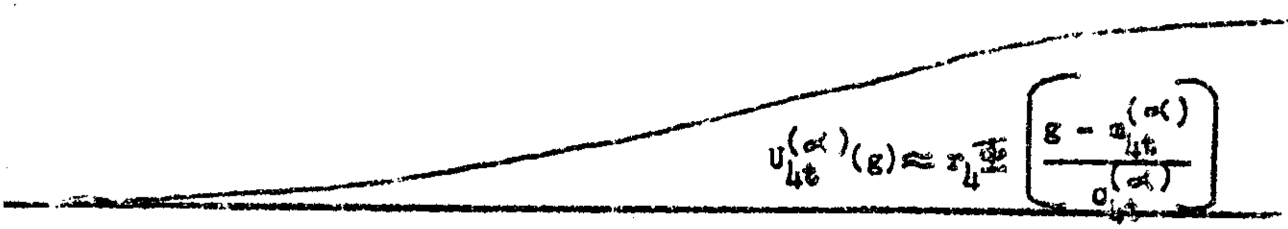
$$U_{2t}^{(\alpha)}(g) \approx r_2 \Phi\left(\frac{g - m_{2t}^{(\alpha)}}{\sigma_{2t}^{(\alpha)}}\right)$$

$g=0$

$U_{2t}^{(\alpha)}(g)$ = increases with the expectation of A_α to <u>get more credit</u> if he asked for it, if he has to take a loss = $-g$.



$g=0$ $U_{3t}^{(\alpha)}(g) \approx r_4 \Phi\left(\frac{g - m_{3t}^{(\alpha)}}{\sigma_{3t}^{(\alpha)}}\right) \approx$ increases with the expectation to be able to "live as usual" <u>without borrowing</u> money if A_α has to take a loss = $-g$.



$$U_{4t}^{(\alpha)}(g) \approx r_4 \Phi\left(\frac{g - m_{4t}^{(\alpha)}}{\sigma_{4t}^{(\alpha)}}\right)$$

$U_{4t}^{(\alpha)}(g) \approx$ increases with A_α's expected ability to improve his physical well-being if he gets a gain = g.

If we now relax the assumption that $P(\tau_g \leq \tau) = \Phi(\frac{\tau - \tau_g}{0})$ and assume that $P(\tau_g \leq \tau) = T_t(\tau)$ it seems at first sight natural to put

$$g_{\alpha}(T) = g \int_{\tau=0}^{\infty} v_{\alpha t}(\tau) \; dT(\tau).$$

but further considerations make it probable that in this case also it may be "safer" to use some kind of "mirror function" $V(v)$ by the calculation of averages over time. We put thus

$$V(v_{\alpha tT}) = \int_{\tau=0}^{\infty} V(v_{\alpha t}(\tau)) \; dT(\tau).$$

and define the value of a gain g receivable at a point of time $\leq t + \tau$ with a probability $T_t(\tau)$ by mean of a formula of the type

$$g_{\alpha tT} = g \cdot v_{\alpha tT}.$$

If we further relax the assumption that A_{α} is alive at point of time $t + T$ and only assume that the estimated probability that A_{α} is alive at $t + \tau$ is $p_{t,\tau}$ and assume that the amount g <u>will not be received</u> at all if A_{α} is "dead" at $t + T$ we have already arrived at a case when we must consider a case when g at $t + T$ have two values (g or 0) with probabilities p_t respective $1 - p_{t\tau}$. This is a special case of the case when we have a two dimensional probability distribution over time and possible gains.

Denote by $F_{\alpha t}(x; \tau)$ the estimated probabilities that A_{α} at $t + \tau$ will have got gains of a total cumulated value x. The goodness $g_{\alpha t} F_{\alpha t}(\cdot,-)$ of this probability distribution can, I think, be evaluated by a formula of the kind

$$U_{\alpha t}(g_{\alpha t}(F_{\alpha t}(\cdot;v)) = \int_{t=0}^{\infty} (\int_{x=-\infty}^{\infty} U_{\alpha t}(x)\ dF_{\alpha t}(x;\tau)\ dV_{\alpha t}(\tau)$$

$$= \int_{\tau=0}^{\infty} U_{\alpha t}(g(F_{\alpha t}(\cdot,\tau))\ dV\ \ _t(\tau).$$

The function $U_{\alpha t}(g(F_{\alpha t}(\cdot;\tau))$ is an indicator of goodness for the distribution of anticipated cumulated gains at point of time $t + \tau$, and $U_{\alpha t}(g_{\alpha t}\ F_{\alpha t}\ (\cdot;v)$ is a weighted mean value of these indicators. The function $V_{\alpha t}(\tau)$ accounts for all discounting and uncertainties in respect to the survival of the subject A_α until $t + \tau$. It can be interpreted as the relative attention paid by A_α at t to events during the period $(t, t + \tau)$ when the attention paid to all future events is used as a unit of measure.

The attention function for business firms is of the geometrical type

$$V_{\alpha t}(\tau) = 1 - e^{-\rho\tau} = \text{business fims weight function}$$

where ρ is the intensity of interest.

Insurance companies use methods equivalent to a assumption that the attention function $V_{\alpha t}(\tau) = 1 - p_{t,\tau} \cdot e^{-\rho\tau} = 1 - e^{-\int_0^{\tau}(\mu_{t,z} + \rho)\ dz}$

where $p_{t,\tau}$ is the probability that the subject A_α will be alive at point of time $t + \tau$ and $\mu_{t,z}$ is the intensity of death at $t + \tau$ for a group of similar persons insured at t.

Of special interest is perhaps the simple cases when

$$U_{\alpha t}(g(F_{\alpha t}(\cdot;\tau)) = U_{\alpha t} + U^{(1)}_{\alpha t} \cdot \tau + U^{(2)}_{\alpha t}\ \tau^2.$$

The mean value $U_{\alpha t}(g(F_{\alpha t}(\cdot,v))$ is then

$$U_{\alpha t}(g(F_{\alpha t}(\cdot,v)) = U_{\alpha t} + U^{(1)}_{\alpha t}\ \bar{\tau}_{\alpha t} + U^{(2)}_{\alpha t}\ (\ \bar{\tau}^2_{\alpha t} + \sigma^2_{\alpha t})$$

$$= \frac{1}{2}\ (U_{\alpha t}(g(F_{\alpha t}(\cdot|\bar{\tau} + \sigma_{\alpha t}) + U_{\alpha t}(g(F_{\alpha t}(\cdot|\bar{\tau}_{\alpha t} - \sigma_{\alpha t}))$$

where

$$\bar{\tau}_{\alpha t} = \int \tau \, d\, V_{\alpha t}(\tau)$$

$$\sigma^2_{\alpha t} = \int (\tau - \bar{\tau})^2 \, dV_{\alpha t}(\tau)$$

The quantity $\bar{\tau}_{\alpha t}$ measures the <u>future mindedness</u> of A_α at t; and $\sigma_{\alpha t}$ <u>the dispersion of attention</u>. In the case $V_{\alpha t}(\tau) = 1 - e^{-\rho\tau}$ we get

$$\tau_{\alpha t} = \frac{1}{\rho} = \sigma_{\alpha t}.$$

If we interpreted α as a plan considered by A_α at t the indicators of goodness arrived at can be interpreted as indicators of goodness for the plan α.