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The Continuity of Multi-valued Functions in Economics [1/]

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1. Introduction

One finds in economics several important examples of variable sets determined by a certain number of parameters. Questions of continuity for a functional relationship of this type arise as soon as one wants to use the concept of such a multi-valued function. In Section 2 a systematic mathematical treatment is presented. The results are applied in Section 3 to the theory of the consumer, in Section 4 to the theory of the producer.

2. Mathematical Study

Only subsets of finite Euclidean spaces will be considered. Two sets $\bar{\mathbb{A}}$ and $\mathbb{A}$ are given and a multi-valued function $A(\bar{a})$ from $\bar{\mathbb{A}}$ to $\mathbb{A}$ which associates with every element $\bar{a}$ of $\bar{\mathbb{A}}$ a certain subset $A(\bar{a})$ of $\mathbb{A}$. k will always denote a positive integer.

Definition 1. The function $A(\bar{a})$ is upper semi-continuous at $\bar{a}_o \in \bar{\mathbb{A}}$ if given any two sequences $(\bar{a}^k)$, (a^k) such that $\bar{a}^k \longrightarrow \bar{a}^o$, $a^k \in A(\bar{a}^k)$ for all k, $a^k \longrightarrow a^o \in \mathbb{A}$, then $a^o \in A(\bar{a}^o)$.

Definition 2. The function $A(\bar{a})$ is lower semi-continuous at $\bar{a}_o \in \bar{\mathbb{A}}$ if given a sequence $(\bar{a}^k)$ such that $\bar{a}^k \longrightarrow \bar{a}^o$, and an element $a^o \in A(\bar{a}^o)$ then there exists a sequence (a^k) such that $a^k \in A(\bar{a}^k)$ for all k, $a^k \longrightarrow a^o$.

Definition 3. The function $A(\bar{a})$ is continuous at $\bar{a}_o \in \bar{\mathbb{A}}$ if it is both l.s.c. and u.s.c. at $\bar{a}_o$.

These concepts occur in Debreu [2] with a different, and less satisfactory, terminology. They can be defined for general topological spaces.

Let $f(\bar{a}, a)$ be a real-valued function defined on the product $\bar{\mathbb{A}} \times \mathbb{A}$. We study the set $M(\bar{a})$ of maximizers of f on $A(\bar{a})$ as a multi-valued function of $\bar{a}$:

$$M(\bar{a}) = \left\{ a' \in A(\bar{a}) \;\middle|\; f(\bar{a}, a') = \max_{a \in A(\bar{a})} f(\bar{a}, a) \right\}$$

$M(\bar{a})$ may be empty.

Theorem 1. If $A(\bar{a})$ is continuous at $\bar{a}^o$ and $f(\bar{a}, a)$ is continuous at every point of $\bar{a}^o \times A(\bar{a}^o)$, then $M(\bar{a})$ is u.s.c. at $\bar{a}^o$.

Consider, according to Definition 1, two sequences $(\bar{a}^k)$, (a^-) $\bar{a}^k \longrightarrow \bar{a}^o$, $a^k \in M(\bar{a}^k)$ for all k, $a^k \longrightarrow a^o \in \mathbb{A}$. We must show that $a^o \in M(\bar{a}^o)$. Since $A(\bar{a})$ is u.s.c. at $\bar{a}^o$, $a^o \in A(\bar{a}^o)$. We have thus only to prove that if $b^o \in A(\bar{a}^o)$, then $f(\bar{a}^o, b^o) \leq f(\bar{a}^o, a^o)$.

Since $A(\bar{a})$ is l.s.c. at $\bar{a}^o$, there is a sequence (b^k) such that $b^k \in A(\bar{a}^k)$ for all k, $b^k \longrightarrow b^o$.

For all k, $f(\bar{a}^k, b^k) \leq f(\bar{a}^k, a^k)$ since $a^k \in M(\bar{a}^k)$. By continuity of f one has, at the limit, the desired inequality.

One may naturally wonder, when $A(\bar{a})$ reduces to a single element what is the relation between u.s. continuity of $A(\bar{a})$ and the ordinary concept of continuity. The answer is given by Theorem 2. Let us first recall:

Definition 4. $A(\bar{a})$ is <u>bounded at $\bar{a}^o$</u> if for some sphere S of center $\bar{a}^o$, $\bigcup_{\bar{a} \in S \cap \bar{\mathbb{A}}} A(\bar{a})$ is bounded.

Theorem 2. <u>If $\mathbb{A}$ is closed, and if the single-valued function $a = \varphi(\bar{a})$ from $\bar{\mathbb{A}}$ to $\mathbb{A}$ is u.s.c. and bounded at $\bar{a}^o$, then φ is continuous at $\bar{a}^o$.</u>

Consider a sequence $(\bar{a}^k)$, $\bar{a}^k \longrightarrow \bar{a}^o$. The set $[\varphi(\bar{a}^k)]$ is bounded, we prove that it has a <u>unique</u> point of accumulation, this will imply continuity at $\bar{a}^o$. Let a' be a point of accumulation of that set; there is a subsequence $(\bar{a}^{k'})$ such that $\varphi(\bar{a}^{k'}) \longrightarrow a'$. Since $\mathbb{A}$ is closed, $a' \in \mathbb{A}$. Obviously $\bar{a}^{k'} \longrightarrow \bar{a}^o$. By u.s. continuity of $\varphi(\bar{a})$ at $\bar{a}^o$, $a' = \varphi(\bar{a}^o)$.

A last result is needed. Let $z = \psi(\bar{a}, a)$ be a single-valued function from $\bar{\mathbb{A}} \times \mathbb{A}$ to a set $\mathbb{Z}$. Define $Z(\bar{a}) = \psi[\bar{a}, A(\bar{a})]$, a multi-valued function from $\bar{\mathbb{A}}$ to $\mathbb{Z}$.

Theorem 3. <u>If $\mathbb{A}$ is closed, $A(\bar{a})$ u.s.c. and bounded at $\bar{a}^o$, $\psi(\bar{a}, a)$ continuous at every point of $\bar{a}^o \times A(\bar{a}^o)$, then $Z(\bar{a})$ is u.s.c. at $\bar{a}^o$.</u>

Consider, according to Definition 1 sequences $(\bar{a}^k)$, (z^k), $\bar{a}^k \longrightarrow \bar{a}^o$, $z^k \in Z(\bar{a}^k)$ for all k, $z^k \longrightarrow z^o \in \mathbb{Z}$. We must show that $z^o \in Z(\bar{a}^o)$.

For every k take an $a^k \in A(\bar{a}^k)$ such that $\psi(\bar{a}^k, a^k) = z^k$. These a^k form a bounded set, extract a converging subsequence $a^{k'} \longrightarrow a^o$ (clearly $\bar{a}^{k'} \longrightarrow \bar{a}^o$, $z^{k'} \longrightarrow z^o$). Since $\mathbb{A}$ is closed, $a^o \in \mathbb{A}$. Since $A(\bar{a})$ is u.s.c. at $\bar{a}^o$, $a^o \in A(\bar{a}^o)$. Since $\psi(\bar{a}, a)$ is continuous at $(\bar{a}^o, a^o)$ $\psi(\bar{a}^{k'}, a^{k'}) \longrightarrow \psi(\bar{a}^o, a^o)$ and so $z^o = \psi(\bar{a}^o, a^o)$.

Most of the substance of these three theorems is given by [2] in a less explicit fashion.

3. Theory of the Consumer

A consumer has to choose a consumption bundle x in some subset X of a finite Euclidean space. X is determined by physical (e.g. non-negativity of some quantities) and physiological contraints. Moreover if p is the price-vector, r his income he is also restricted by $p.x \leq r$. We denote

$$X(p, r) = \left\{ x \in X \;\middle|\; p \,.\, x \leq r \right\}.$$

The choice of x in X(p, r) is made so as to maximize a real-valued utility function u(x) defined on X. In this section we always assume that: X is closed and convex, X is not contained in the half-space $p^o.x \geq r^o$, u(x) is continuous on X.

Arrow-Debreu [1] have proved:

(3.1) X(p, r) is continuous at (p^o, r^o)

The set of optimal consumption bundles for (p, r) (set of maximizers of u(x) on X(p, r)) is denoted by M(p, r). A straight forward application of theorem 1 (Section 2) then gives [with (3.1)].

(3.2) M(p, r) is u.s.c. at (p^o, r^o).

From (3.2) follow the answers to two other unsettled questions of consumer's theory. If assumptions (strict quasi-concavity) on u(x) are such that there is always at most one optimal consumption bundle, then M(p, r) is single-valued and theorem 2 (Section 2) gives [with (3.2)].

(3.3) If M(p, r) is single-valued and bounded at (p^o, r^o), then M(p, r) is continuous at (p^o, r^o).

The paper [4] by Houthakker is based on the conjecture that (3.3) holds. In the general case where M(p, r) is not necessarily single-valued, one can define u(p, r) as u(x) for some $x \in M(p, r)$ (all these x give the same value for u(x)). u(p, r) is called the "indirect" utility function (following Houthakker); this concept has been introduced by Hotelling [3] p. 594 and Roy [5]

By theorem 3 (Section 2), (3.2), and theorem 2 (Section 2),

(3.4) If M(p, r) is bounded at (p^o, r^o), then u(p, r) is continuous at (p^o, r^o).

4. Theory of the Producer

A producer has to choose an input-output combination y in some set Y of technological possibilities so as to maximize the profit p.y. The set of optimal y for p(set of maximizers of p.y on Y) is denoted by M(p).

In this section we always assume that Y is closed.

The constraining set Y is here a constant and therefore trivially a continuous multi-valued function. Theorem 1 (Section 2) then gives

(4.1) M(p) is an u.s.c. function of p.

By theorem 2 (Section 2) and (4.1).

(4.2) If M(p) is single-valued and bounded at p^o, then M(p) is continuous at p^o.

Call π(p) the maximum of profit for a given p. By theorem 3 (Section 2) and (4.1):

(4.3) If M(p) is bounded at p^o, then π(p) is continuous at p^o.

Footnote

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References

[1] Arrow-Debreu, "Existence of an Equilibrium for a Competitive Economy", forthcoming.

[2] Debreu, "A Social Equilibrium Existence Theorem", Proceed. Nat. Acad. Sc., Volume 38, 1952, pp. 886-893.

[3] Hotelling, "Edgeworth's Taxation Paradox and the Nature of Demand and Supply Functions", Journal of Political Economy, Vol. 40, October, 1932, pp. 577-616.

[4] Houthakker, "La forme des Courbes d'Engel", Cahiers du Séminaire d'Econométrie, Vol. 2, forthcoming.

[5] Roy, De l'Utilité, Paris, Hermann, 1942.