The Continuity of Multi-valued Functions in Economics

Gerard Debreu
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1. Introduction

One finds in economics several important examples of variable sets determined by a certain number of parameters. Questions of continuity for a functional relationship of this type arise as soon as one wants to use the concept of such a multi-valued function. In Section 2 a systematic mathematical treatment is presented. The results are applied in Section 3 to the theory of the consumer, in Section 4 to the theory of the producer.

2. Mathematical Study

Only subsets of finite Euclidean spaces will be considered. Two sets  and  are given and a multi-valued function  from  to  which associates with every element  of  a certain subset  of , will always denote a positive integer.

Definition 1. The function  is upper semi-continuous at  if given any two sequences  and  such that  and  for all  and , then .
Definition 2. The function $A(\bar{a})$ is lower semi-continuous at $\bar{a}_o \in \bar{A}$ if given a sequence $(\bar{a}^k)$ such that $\bar{a}^k \rightarrow \bar{a}_o$, and an element $a^o \in A(\bar{a}^o)$, then there exists a sequence $(a^k)$ such that $a^k \in A(\bar{a}^k)$ for all $k$, $a^k \rightarrow a^o$.

Definition 3. The function $A(\bar{a})$ is continuous at $\bar{a}_o \in \bar{A}$ if it is both l.s.c. and u.s.c. at $\bar{a}_o$.

These concepts occur in Debreu [2] with a different, and less satisfactory, terminology. They can be defined for general topological spaces.

Let $f(\bar{a}, a)$ be a real-valued function defined on the product $\bar{A} \times \bar{A}$. We study the set $M(\bar{a})$ of maximizers of $f$ on $A(\bar{a})$ as a multi-valued function of $\bar{a}$:

$$M(\bar{a}) = \left\{ a^i \in A(\bar{a}) \mid f(\bar{a}, a^i) = \max_{a \in A(\bar{a})} f(\bar{a}, a) \right\}$$

$M(\bar{a})$ may be empty.

Theorem 1. If $A(\bar{a})$ is continuous at $\bar{a}_o$ and $f(\bar{a}, a)$ is continuous at every point of $\bar{a}_o \times A(\bar{a}_o)$, then $M(\bar{a})$ is u.s.c. at $\bar{a}_o$.

Consider, according to Definition 1, two sequences $(\bar{a}^k), (\bar{a}^r) \bar{a}^k \rightarrow \bar{a}_o$, $\bar{a}^r \rightarrow \bar{a}_o \in \bar{A}$. We must show that $\bar{a}_o \in M(\bar{a}_o)$. Since $A(\bar{a})$ is u.s.c. at $\bar{a}_o$, $\bar{a}_o \in A(\bar{a}_o)$. We have thus only to prove that if $b^o \in A(\bar{b}_o)$, then $f(\bar{a}_o, b^o) \leq f(\bar{a}_o, a^o)$.

Since $A(\bar{a})$ is l.s.c. at $\bar{a}_o$, there is a sequence $(b^k)$ such that $b^k \in A(\bar{a}^k)$ for all $k$, $b^k \rightarrow b^o$.

For all $k$, $f(\bar{a}^k, b^k) \leq f(\bar{a}^k, a^k)$ since $a^k \in M(\bar{a}^k)$. By continuity of $f$ one has, at the limit, the desired inequality.

One may naturally wonder, when $A(\bar{a})$ reduces to a single element what is the relation between u.s. continuity of $A(\bar{a})$ and the ordinary concept of continuity. The answer is given by Theorem 2. Let us first recall:
Definition 1. \( A(\bar{a}) \) is bounded at \( \bar{a}^0 \) if for some sphere \( S \) of center \( \bar{a}^0 \),

\[
\bigcup_{\bar{a} \in S \cap \bar{A}} A(\bar{a}) \text{ is bounded.}
\]

Theorem 2. If \( \bar{a} \) is closed, and if the single-valued function \( a = \varphi(\bar{a}) \)
from \( \bar{A} \) to \( \bar{A} \) is u.s.c. and bounded at \( \bar{a}^0 \), then \( \varphi \) is continuous at \( \bar{a}^0 \).

Consider a sequence \( (\bar{a}^k) \), \( \bar{a}^k \rightarrow \bar{a}^0 \). The set \( \{ \varphi(\bar{a}^k) \} \) is bounded, we prove that it has a unique point of accumulation, this will imply continuity at \( \bar{a}^0 \). Let \( a' \) be a point of accumulation of that set; there is a subsequence \( (\bar{a}^k') \) such that \( \varphi(\bar{a}^k') \rightarrow a' \). Since \( \bar{A} \) is closed, \( a' \in \bar{A} \). Obviously \( \bar{a}^k' \rightarrow \bar{a}^0 \). By u.s. continuity of \( \varphi(\bar{a}) \) at \( \bar{a}^0 \), \( a' = \varphi(\bar{a}^0) \).

A last result is needed. Let \( \bar{z} = \psi(\bar{a}, a) \) be a single-valued function from \( \bar{A} \times \bar{A} \) to a set \( \bar{Z} \). Define \( \bar{Z}(\bar{a}) = \psi(\bar{a}, A(\bar{a})) \), a multi-valued function from \( \bar{A} \) to \( \bar{Z} \).

Theorem 3. If \( \bar{A} \) is closed, \( A(\bar{a}) \) u.s.c. and bounded at \( \bar{a}^0 \), \( \psi(\bar{a}, a) \) continuous at every point of \( \bar{a}^0 \times A(\bar{a}^0) \), then \( \bar{Z}(\bar{a}) \) is u.s.c. at \( \bar{a}^0 \).

Consider, according to Definition 1 sequences \( (\bar{a}^k') \), \( (z^k) \), \( \bar{a}^k \rightarrow \bar{a}^0 \), \( z^k \in \bar{Z}(\bar{a}^k) \) for all \( k \), \( z^k \rightarrow z^0 \in \bar{Z} \). We must show that \( z^0 \in \bar{Z}(\bar{a}^0) \).

For every \( k \) take an \( a^k \in A(z^k) \) such that \( \psi(z^k, a^k) = z^k \). These \( a^k \) form a bounded set, extract a converging subsequence \( a^{k_1} \rightarrow a^0 \) (clearly \( a^{k_1} \rightarrow a^0 \), \( z^{k_1} \rightarrow z^0 \)). Since \( \bar{A} \) is closed, \( a^0 \in \bar{A} \). Since \( A(\bar{a}) \) is u.s.c. at \( \bar{a}^0 \), \( a^0 \in A(\bar{a}^0) \). Since \( \psi(\bar{a}, a) \) is continuous at \( (\bar{a}^0, a^0) \), \( \psi(\bar{a}^{k_1}, a^{k_1}) \rightarrow \psi(\bar{a}^0, a^0) \) and so \( z^0 = \psi(\bar{a}^0, a^0) \).

Most of the substance of these three theorems is given by \( [2] \) in a less explicit fashion.
3. Theory of the Consumer

A consumer has to choose a consumption bundle \( x \) in some subset \( X \) of a finite Euclidean space. \( X \) is determined by physical (e.g., non-negativity of some quantities) and physiological constraints. Moreover if \( p \) is the price-vector, \( r \) his income he is also restricted by \( p \cdot x \leq r \). We denote

\[
X(p, r) = \{ x \in X \mid p \cdot x \leq r \}.
\]

The choice of \( x \) in \( X(p, r) \) is made so as to maximize a real-valued utility function \( u(x) \) defined on \( X \). In this section we always assume that \( X \) is closed and convex, \( X \) is not contained in the half-space \( p^0 \cdot x \geq r^0 \), \( u(x) \) is continuous on \( X \).

Arrow-Debreu [1] have proved:

(3.1) \( X(p, r) \) is continuous at \((p^0, r^0)\).

The set of optimal consumption bundles for \((p, r)\) (set of maximizers of \( u(x) \) on \( X(p, r) \)) is denoted by \( \mathcal{W}(p, r) \). A straightforward application of theorem 1 (Section 2) then gives [with (3.1)].

(3.2) \( \mathcal{W}(p, r) \) is u.s.c. at \((p^0, r^0)\).

From (3.2) follow the answers to two other unsettled questions of consumer's theory. If assumptions (strict quasi-concavity) on \( u(x) \) are such that there is always at most one optimal consumption bundle, then \( \mathcal{W}(p, r) \) is single-valued and theorem 2 (Section 2) gives [with (3.2)].

(3.3) If \( \mathcal{W}(p, r) \) is single-valued and bounded at \((p^0, r^0)\), then \( \mathcal{W}(p, r) \) is continuous at \((p^0, r^0)\).

The paper [4] by Houthakker is based on the conjecture that (3.3) holds. In the general case where \( \mathcal{W}(p, r) \) is not necessarily single-valued, one can define \( u(p, r) \) as \( u(x) \) for some \( x \in \mathcal{W}(p, r) \) (all these \( x \) give the same value for \( u(x) \)). \( u(p, r) \) is called the "indirect" utility function (following Houthakker); this concept has been introduced by Hotelling [3] p. 99, and Roy [5]
By theorem 3 (Section 2), (3.2), and theorem 2 (Section 2),

(3.1) If \( M(p, r) \) is bounded at \( (p^0, r^0) \), then \( u(p, r) \) is continuous at \( (p^0, r^0) \).

\[ h. \text{ Theory of the Producer} \]

A producer has to choose an input-output combination \( y \) in some set \( Y \) of technological possibilities so as to maximize the profit \( p.y \). The set of optimal \( y \) for \( p \) (set of maximizers of \( p.y \) on \( Y \)) is denoted by \( M(p) \).

In this section we always assume that \( Y \) is closed.

The constraining set \( Y \) is here a constant and therefore trivially a continuous multi-valued function. Theorem 1 (Section 2) then gives

(4.1) \( M(p) \) is an u.s.c. function of \( p \).

By theorem 2 (Section 2) and (4.1);

(4.2) If \( M(p) \) is single-valued and bounded at \( p^0 \), then \( M(p) \) is continuous at \( p^0 \).

Call \( \Pi(p) \) the maximum of profit for a given \( p \). By theorem 3 (Section 2) and (4.1);

(4.3) If \( M(p) \) is bounded at \( p^0 \), then \( \Pi(p) \) is continuous at \( p^0 \).
Footnote

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References


