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Conditions under which Communication is Superfluous[1/]

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1. Consider the expression (often mentioned in previous discussions)

$$U = \int_x u(a,x)dF(x)$$

where $a = (a_1,\dots,a_n)$ denotes actions, $x = (x_1,\dots,x_n)$ denotes observables, and $F(x)$ is a distribution. Assume a and x to be real numbers and assume the existence of second derivatives

$$\frac{\partial^2 u}{\partial a_i \, \partial a_j} = u_{ij} \quad \text{and of} \quad \frac{\partial^2 u}{\partial a_i \, \partial x_j} = v_{ij}$$

Let $\hat{a} = (\hat{a}_1,\dots,\hat{a}_n)$ be the value of the vector a that maximizes U for a fixed x. Then, as x varies, we can trace a set of functions, $\hat{a}_i = \alpha_i(x)$,

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$i = 1,\ldots,n$. Since $dF(x)$ is non-negative, these functions maximize U. They give the best actions of the n partners if every one is informed of all the x's. We want to find the conditions under which $\alpha_i(x)$ depends on x_i only. When this is the case, we may say that "communication is superfluous" or "a completely decentralized network is optimal."

Sections 2, 3, 4 suggest a necessary condition for superfluity of communications, depending on properties of u, for any distribution F. Section 5 consists in mentioning a sufficient condition on F, for any u. The complete problem is, of course, to find necessary and sufficient conditions on both u and F.[2/]

2. For fixed x, and for $a = \hat{a}$, we have

$$\frac{\partial u}{\partial a_i} = 0, \; i = 1,\ldots, n.$$

This is a set of identities in x. Differentiating with respect to x,

(2.1) $$\sum_k u_{ik} \frac{d\hat{a}_k}{dx_j} + v_{ij} = 0.$$

In matrix notation, with $[u_{ij}] = U$ (a negative definite symmetric matrix), $[v_{ij}] = v$ and $\left[\frac{d\hat{a}_i}{dx_j}\right] = A$, we obtain

(2.2) $$U^{-1} V = -A \;.$$

We require that

(2.3) $$\frac{d\hat{a}_k}{dx_j} = 0 \text{ whenever } k \neq j \;;$$

this implies by (2) that

(2.4) $$U^{-1} V \text{ is diagonal.}$$

3. A verbal interpretation is obtained by combining (1) and (3) ;

2/ This discussion may perhaps open the way to finding the conditions under which, more generally, $\alpha_i(x)$ depends on a given subset $x^{(i)}$ of x, so that decentralization is not complete, and is characterized by a given set $X = (x^{(1)},\ldots, x^{(n)})$ which gives the state of information after communication.

$$(3.1) \qquad \frac{u_{ij}}{v_{ij}} = - \frac{d\hat{a}_j}{dx_j} ;$$

that is, the ratio u_{ij} / v_{ij} depends on j only, but not on i. In words:

A necessary condition for communication to be superfluous is the following: in the neighborhood of the set of best actions, the marginal effect of a member's action is influenced by a small change in the action of any of his partners in proportion to its being influenced by a small change in the external variable observed by that partner.

4. In particular, it appears that the condition is satisfied if u(a, x), appropriately normalized, belongs to the following class of quadratic forms (studied by Radner):

$$U = \sum_{i,j} (a_i - x_i)\, q_{ij}\, (a_j - x_j),$$

in which case simply $\hat{a} = x$, so that A is the identity matrix, and U = -V. Moreover these matrices are constant, $U = [q_{ij}]$; so that the proviso "in the neighborhood of the set of best actions" can be suppressed.

5. <u>A remark on the role of F</u>. The conditions (2.4) or (3.1) depend on properties of u only and not on those of F. A different problem is presented by the properties of F necessary or sufficient to make communications superfluous, for any payoff function u. For example, here is a sufficient condition: if there is a random set Y such that, for every i, there is a one-to-one correspondence between the values of x_i and the elements of Y, then communication is superfluous.[3] This condition is proposed on intuitive grounds: the observer of x_i will be able to infer, <u>via</u> y, the values of all other elements of x. Perhaps the formalization and modification of this condition may spark further ideas on the role of F, and also on the joint role of F and u, in determining optimal networks.

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[3] A particular case is that of F being normal, with perfect correlation for each pair x_i, x_j.

6. Theorem 1.

If the pay-off function has the property:

$$u(\vec{x}, \vec{a}) = u(\vec{x}, \vec{a}_1, \vec{a}_2) = u_o(\vec{x}, \vec{a}_2) - u_1(\vec{x}, \vec{a}_1, \vec{a}_2);$$

where $\vec{a} = (\vec{a}_1, \vec{a}_2)$, $\vec{a}_1 = (a_{11}, \ldots, a_{1n_1})$, $\vec{a}_2 = (a_{21}, \ldots, a_{2n_2})$.

$$u_1(\vec{x}, \vec{a}_1, \vec{a}_2) \geq 0$$

$$u_1(\vec{x}, \vec{a}_1 \in A_o(\vec{x}_1), \vec{a}_2) = 0$$

and the set of functions $\vec{a}_1 = \alpha_1(\vec{x}_1) \in A_o(\vec{x}_1)$ is not empty for any possible $(\vec{x}, \vec{a}_2)$, then any $\alpha_1(\vec{x}_1) \in A_o(\vec{x}_1)$ define an optimal rule of action for the decision-maker controlling $\vec{a}_1$, regardless of the distribution of $(\vec{x}_2, \vec{a}_2)$. Information about $(\vec{x}_2, \vec{a}_2)$ is then superfluous for the decision-maker in question.

This theorem follows from the fact that if any rule of action $\vec{a}_1 = \alpha_1(\vec{x})$ which does not belong to the set $A_o(\vec{x}_1)$ is chosen the expected value of $u(\vec{x}, \alpha_1(\vec{x}), \alpha_2(\vec{x}))$ is smaller than the expected value of $u_o(\vec{x}, \alpha_2(\vec{x}))$ corresponding to a rule $\vec{a}_1 = \alpha_1(\vec{x}_1) \in A(\vec{x}_1)$.

7. Theorem 2.

If the set $A_o(\vec{x}_1)$ is empty but there exists a non empty set of functions $\alpha_c(\vec{x}_1) \in A_c(\vec{x}_1)$ such that for every possible $(\vec{x}, \vec{a})$

$$- u_1(\vec{x}, \vec{a}) + u_1(\vec{x}, \vec{a}_1 = \alpha_c(\vec{x}_1) \in A_c(\vec{x}_1), \vec{a}_2) \leq C$$

where C is the minimum cost of informing the decision-maker controlling $\vec{a}_1$ about some property of $(\vec{x}_2, \vec{a}_2)$ and of its use in a process of thinking that might result in a better choice of $\vec{a}_1$ than any $\alpha_c(\vec{x}_1) \in A_c(\vec{x}_1)$, then the rules of action $\vec{a}_1 = \alpha_c(\vec{x}_1)$ are all better than any rule of action which is dependent on information about $(\vec{x}_2, \vec{a}_2)$.

The superfluity of communication in this case follows from the fact that communication costs more than the increase in the payoff that could be got by a departure from a rule which does not make use of the information that could be obtained.

The case when $(\vec{a}_2, \vec{x}_2)$ is a uniquely determined function of $\vec{x}_1$ known by the decision-maker is a special case of theorem 1.