COWLES COMMISSION DISCUSSION PAPER: ECONOMICS NO. 2077

Structural and Operational Communication Problems in Teams. II

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May 13, 1953

Contents

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1. Research undertaken by the Cowles Commission for Research in Economics under Contract Nonr-358(01), NR 047-006 with the Office of Naval Research.

I. Introduction.

This is the second of a pair of papers dealing with two important aspects of the problem of organization in teams.[1] [2] These two aspects — called structure and operation — have been incorporated into a mathematical model, which is described and illustrated by Marschak and Radner in the first paper [1]. We shall now take a look at two types of problems which fall into the general framework of that model.

The first is concerned with splitting up a team — for the purposes of communication and decision making — into disjoint groups. This might be called a problem of decentralization. (Part II). The second problem concerns a situation in which the team consists of several observers reporting to a single individual (or central office) who then acts on the basis of these reports. (Part III).

The reader is referred to Part 1 of [1] for a discussion of the relationship between these two types of problems and the general model. A summary of the notation introduced there is given below for reference, and this notation will be used in what follows without further explanation.

Summary of Notation for General Model

$M = 1, \ldots, m$

$a = (a_1, \ldots, a_m)$, action variables

$x = (x_1, \ldots, x_m)$, observable random variables with distribution $O(x)$.

$u(a, x)$, gross payoff

$S = \{S_i\}_{i \in M}$, $S_i \subseteq M$, the structure

$x^{(i)} = \{x_j^{(i)}\}, j \in S_i$

1. For a general discussion of the concept of a team as a special type of organization see Marschak, [2].

2. I wish to thank J. Marschak and M. Beckmann for many helpful discussions concerning the material of this paper.

$y^{(i)} = \omega^{(i)}(x^{(i)})$, the information about x on which a_i is based, where

$$\omega^{(i)} = \{\omega_j^{(i)}\}_{j \in S_i} \qquad \omega_j^{(i)}(x_j^{(i)}) = \begin{cases} x_j^{(i)} \\ \theta \end{cases}$$

$\omega = \{\omega^{(i)}\}_{i \in M}$, the operation

$\alpha_i = \alpha_i(y^{(i)}) \quad \alpha = \{\alpha_i\}$, the rule of action

$c(S)$, cost of structure S

$C(\omega) = \sum_{i \in M} \sum_{j \in S_i} C_{ij}$, cost of operation per unit of activity, where

$$C_{ij} = \begin{cases} c_{ij} \\ 0 \end{cases} \text{ as } \omega_j^{(i)}(x_j^{(i)}) = \begin{cases} x_j^{(i)} \\ \theta . \end{cases}$$

$v = u(\alpha(x), x) - c(S) - C(\omega)$, net payoff

A <u>full operation</u> is defined by:

$$\omega_j^{(i)}(x_j^{(i)}) = x_j^{(i)}, \text{ for all } x_j^{(i)} \text{ and all } i, j.$$

A <u>pure structure problem</u> is defined as one in which only full operations are considered.

II. A Problem of Decentralization.

In this section we shall consider the pure structure problem [1] for a team whose chief characteristic is that the only allowable structures are those which split the team -- for the purposes of information and decision making -- into disjoint groups. The assumptions made are:

1) The observable random variables x_i (real) are independent, with means μ_i and variances σ_i^2 .

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1. Cf. [1], 1.3.1.

2) The actions a_i are real numbers, $-\infty < a_i < \infty$.

3) The gross payoff $u(a, x)$ is any quadratic function in the a_i and x_i such that for every x, $u(a, x)$ has a unique maximum with respect to a.

4) The structure $S = \{S_i\}$ is restricted by the conditions:

(i) either $S_i = S_j$ or $S_i \cap S_j = 0$

(ii) $i \in S_i$, for all i

This implies that the variables are partitioned i.e. there are disjoint sets M_n whose union is M such that $S_i = M_n$ if and only if $i \in M_n$. Any admissible structure S, then is defined by the partition M_n.

5) The operation costs c_{ij} are zero for all i and j, and hence for any structure the corresponding full operation is used.

In interpreting what follows it is useful to visualize the team as consisting of m members, with the i^{th} member observing X_i and performing a_i. These m members are partitioned into disjoint groups M_n, such that each one communicates the value of his x_i to all the others in his group, and to no one else.

For any structure, we wish to find the best rule of action and the corresponding expected payoff.

Using vector and matrix notation we may write the gross payoff as

$$(1) \qquad u(a, x) = a\,Q\,a' - 2\,x\,R\,a' - 2\,q\,a'$$

where Q and R are matrices, Q is symmetric and q is a constant vector. (Possible terms in x alone need not be considered, since these would not be affected by changes in structure or rule of action.)

Assumption (3) above implies that Q is negative definite. The transformation

$$\begin{cases} a = b + (\mu R + q)\,Q^{-1} \\ x = z + \mu \end{cases}$$

brings the payoff (1) into the form

$$b\,Q\,b' - 2\,z\,R\,b' + \text{terms in } y \text{ alone} + \text{constant terms}$$

Hence there is no loss of generality in taking equation (1) to be:

(1')
$$u(a, x) = a\,Q\,a' - 2\,x\,R\,a'$$
$$\text{where } \mathcal{E}x = 0$$ [1]

If we make the further transformation

$$x_i = \sigma_i\, z_i$$

then (1') becomes

<u>Gross Payoff</u>:

(1")
$$w(a, z) = u(a, x) = a\,Q\,a'' - 2\,z\,T\,a'$$
$$\text{where} \begin{cases} \mathcal{E}z = 0 \\ \mathrm{Var}(z_i) = 1, \quad \mathcal{E}z_i\,z_j = 0 \text{ if } i \neq j \\ T = D\,R \\ D \text{ is diagonal with } d_{ii} = \sigma_{ii} \end{cases}$$

Before stating the main result we need the following notation: For a given structure $S = \{M_n\}$ let Q_n denote the matrix $((q_{ij}))_{i,j \in M_n}$ of those elements q_{ij} of Q such that i and j are in M_n, and similarly for R_n, T_n and D_n. Also, let z^n denote the vector $(z_j)_{j \in M_n}$ of those components z_j of z such that $j \in M_n$, and similarly for x^n, a^n and α^n. Then subject to the various assumptions made up to now, we have the

<u>Theorem</u>. For any structure S, the best rule of action $\hat{\alpha}$ is given by

(2)
$$\hat{\alpha}_n(x^n) = x^n\,R_n\,Q_n^{-1}$$

and the corresponding (maximum) expected <u>gross</u> payoff is

(3)
$$U(S) = -\sum_n \mathrm{tr}\,(D_n\,R_n\,Q_n^{-1}\,R_n^1\,D_n')$$ [2]

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1. $\mathcal{E}x$ = expected value of x, Var x = variance of x.

2. tr(A) = the trace of A.

Proof:

If the structure $S = \{M_k\}$, then the expected gross payoff U for any α is

$$U(\alpha) = \sum_k \sum_{i,j \in M_k} q_{ij}\, \mathcal{E}[\alpha_i(z^k)\alpha_j(z^k)]$$

$$+ \sum_{\substack{n,k \\ n \neq k}} \sum_{i \in M_n} \sum_{j \in M_k} q_{ij}\, \mathcal{E}[\alpha_i(z^k)]\mathcal{E}[\alpha_j(z^k)]$$

$$- 2 \sum_{n,k} \sum_{i \in M_n} \sum_{j \in M_k} t_{ij}\, \mathcal{E}[z_i\, \alpha_j\, (z^k)]$$

Notice that we have made use of the fact that the expected value of a product of independent random variables is equal to the product of their expectations.

Applying the Euler conditions for an extremum of U with respect to [cf. [3]):

(4) $$\sum_{i \in M_k} q_{ij}\, \alpha_i(z^k)$$

$$+ \sum_{n \neq k} \sum_{i \in M_n} q_{ij}\, \mathcal{E}[\alpha_i(z^n)]$$

$$- \sum_{i \in M_k} t_{ij}\, z_i$$

$$- \sum_{n \neq k} \sum_{i \in M_n} t_{ij}\, \mathcal{E}[z_i] = 0$$

Noting that the last term of (4) is zero, we can rewrite (4) as:

(4a) $$\sum_{i \in M_k} q_{ij}\, \alpha_i(z^k) = \sum_{i \in M_k} t_{ij}\, z_i$$

$$- \sum_{n \neq k} \sum_{i \in M_n} q_{ij}\, \mathcal{E}[\alpha_i(z^n)],\ (j \in M_k)$$

In matrix notation this becomes

$$\alpha^k\, (z^k)\, Q_k = z^k\, T_k - b^k$$

where b^k is the last term of (4a), a constant vector independent of α^k. Since Q is negative definite, Q_k is non singular, and

(4b) $$\alpha^k(z^k) = z^k T_k(Q_k)^{-1} - b^k(Q_k)^{-1}$$

$$\mathcal{E}[\alpha^k(z^k)] = -b^k(Q_k)^{-1}$$

(4c) $$\mathcal{E}[\alpha^k(z^k)]\, Q_k = -b^k$$

As a matter of fact, $\mathcal{E}[\alpha^k(z^k)]=0$, as we will now see. Substituting the definition of b^k into (4c) gives

$$\sum_{i\in M_k} q_{ij}\, \mathcal{E}[\alpha_i(z^k)] = -\sum_{n\neq k}\ \sum_{i\in M_n} q_{ij}\, \mathcal{E}[\alpha_i(z^n)], \text{ all } j\in M_k$$

(4d) $$\sum_k \sum_{i\in M_k} q_{ij}\, \mathcal{E}[\alpha_i(z^k)] = 0 .$$

Since Q is non singular, (4d) implies that $\mathcal{E}[\alpha_i(z^k)]=0$ for all i, and hence $b^k=0$ for all k. Thus the optimal α, given by (4b) can now be written

(4e) $$\alpha^k(z^k) = z^k T_k(Q_k)^{-1}$$

which establishes equation (2). Substituting this in equation (4) gives

$$U(S) = \sum_n \mathcal{E}_{z_n}\{-Z^n T_n Q_n^{-1} T_n' Z^{n'}\}$$

which, again because of the independence of the z_i, reduces to equation (3). QED

In attempting to interpret our result, the first thing we might notice is that if R=Q, then

$$U(S) = -\sum_n \operatorname{tr}(D_n Q_n D_n)$$

(5) $$= -\sum_i \sigma_i^2 q_{ii}$$

which is independent of S. In particular, since the structure S^1 in which each S_i consists of i only (i.e. each action a_i is based upon x_i only) is presumably the cheapest of all the allowable structures, it will in this case always be optimal. In terms of our concrete (!) example, if R=Q, there is no advantage in having communication between members of the team.

In fact the case R=E Q, where E is diagonal, is the only one in which no other structure has a larger expected gross payoff than $U(S^1)$. [1] This is easily seen once it is noted that for any fixed x, the best action a is

$$\hat{a} = x R Q^{-1}$$

i.e. $\hat{a}_i$ depends only on x_i, only if $R Q^{-1}$ is diagonal. We summarize this in the

Corollary 1. There is no (gross) advantage in communication if and only if $R \neq E Q$ for all diagonal E.

Another case of special interest is the one $Q = -I$ (I = the identity matrix) for in this case the gross advantage of communication in different structures assumes a very simple form.

The condition $Q = -I$ means that there is no "interaction" between the actions of different members. Thus, if $Q = -I$, we have from (3):

$$U(S) = \sum_n \mathrm{tr}\,(D_n R_n R_n' D_n)$$

$$= \sum_n \sum_{i,j \in M_n} \sigma_i^2 r_{ij}^2$$

$$U(S^o) = \sum_{i \in M} \sigma_i^2 r_{ii}^2$$

(6) $$U(S) - U(S^o) = \sum_n \sum_{\substack{i,j \in M_n \\ i \neq j}} \sigma_i^2 r_{ij}^2$$

This last equation has a simple intuitive interpretation. Since r_{ij} is proportional to $-\frac{\partial^2 u}{\partial a_i \; \partial x_j}$, we might say that the advantage of any "communication

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1. Strictly speaking this is true only if $\sigma_i^2 > 0$ for all i, that is if none of the x_i are constant with probability one.

group" M_n is measured by the sum of squares of "interactions" between the actions of the group members and the observable variables of other members of the same group, weighted by the variances of those observables.

In summary, we have:

1) Determined the best rule of action and expected gross payoff for any "pattern of decentralization."

2) Found a necessary and sufficient condition for there to be no (gross) advantage in any degree of centralization.

3) Given an intuitive interpretation of the best expected payoff for the case in which there is no interaction between the actions of different team members.

III. The n-Observer, 1-Actor Team.

1. Introduction. Consider a team in which there are several observers reporting to a central office, which then acts on the basis of these reports. If observation and/or communication is costly it may not be worthwhile to observe all the relevant variables, nor to have the observers always report the information they possess.

The formal specification of this problem is given in [1] section 1.3.4. It will be convenient to use slightly different notation here; thus our problem is defined by:

(a) $x = (x_1, \ldots, x_n)$, the (real) vector of observable random variables, with joint distribution $\Phi(x)$, means μ_i and covariances σ_{ij}.

(b) a, the action variable. We think of the observers as taking no "action," while the single "actor" makes no observations.

(c) $u(a, x)$, the gross payoff.

(d) Let N be the set of integers 1, ..., n. Then any admissible structure is defined by a subset $S \subseteq N$, which determines those variables which are to be observed.

(e) The operation is determined by $\omega = \{\omega_i\}_{i \in S}$, where $\omega_i(x_i) = 1$ or 0, according as x_i is reported or not.

(f) For each structure S there is a cost c(S). The activity of the team is conceived of as taking place in discrete units, each unit consisting of a set of observations, a set of reports, an action and a payoff.

In addition to the structure cost c(S) there is a cost c per variable reported per unit of activity.

If the variables x_i, $i \in I \subset S$ are actually reported, then some action $\alpha_I(x^I)$ will be taken, where $x^I = \{x_i\}_{i \in I}$. If we let $\alpha = \{\alpha_I\}_{I \subset S}$, then the triple (S, ω, α) completely defines the constitution of the team. We wish to find that constitution which maximizes the expected net payoff per unit of activity.

In this section we will consider two types of action and payoff. The first, which will be designated Case 1, is one in which the loss due to not knowing the exact value of x can be measured by a quadratic "distance" between the true x and the estimated one. Precisely, $a = (a_1, \ldots, a_n)$ is a real vector and

(1) $\qquad u(a, x) = -(a - x)\, Q(a - x)'$, $Q = ((q_{ij}))$ positive definite.

In Case 2, the action a can take on only two values, 1 or 0, say, and the net payoff is of the form

(2) $\qquad u(a, x) = a \sum_{i=1}^{n} w_i(x_i)$.

In both cases the problem is solved most readily if the x_i are independently distributed. Dependance is treated here only under special assumption of normality, and even then only the pure structure problem is solved in each case. We shall see that the important property of homoscedasticity is the key to the solution when the x_i are normal.

2. Case 1. Quadratic Payoff

2.1 Independent x_i

2.1.1 Theorem. For any given structure S, the best operation w is given by:

(3a) $$\omega_i(x_i = \begin{smallmatrix}1\\0\end{smallmatrix} \text{ according as } \left|x_i - \hat{a}_i\right| \gtrless \sqrt{\frac{c}{q_{ii}}} \text{ for } i \in S$$

where $\hat{a}_i$ is determined by

(3b) $$\int_{-\sqrt{\frac{c}{q_{ii}}}}^{\sqrt{\frac{c}{q_{ii}}}} t d\Phi_i(t + \hat{a}_i) = 0$$

If Φ_i is absolutely continuous and unimodal then

(3d) $$\hat{a}_i = \mu_i$$

The maximum net payoff, given S is

(3c) $$V(S) = -\sum_{i \in S} \int_{\hat{a}_i - \sqrt{\frac{c}{q_{ii}}}}^{\hat{a}_i + \sqrt{\frac{c}{q_{ii}}}} [q_{ii}(x_i - \hat{a}_i)^2 - c] \, d\Phi_i(x_i)$$

$$- k c - \sum_{j \notin S} q_{jj} \sigma_j^2 - c(S)$$

The best rule of action $\hat{\alpha}$ is given by

$$(3e) \quad \left\{ \begin{array}{l} \hat{\alpha}_i(x_i) = x_i \quad \text{if } \omega_i(x_i) = 1 \\ \hat{\alpha}_i = \hat{a}_i \quad \text{if } \omega_i(x_i) = 0 \\ \hat{\alpha}_j = \mu_j, \qquad j \notin S. \end{array} \right\} \quad i \in S$$

Proof:

Suppose the structure S is fixed, and the variables are renumbered so that $S = \{1, 2, \ldots, k\}$. For any subset N of $M = \{1, \ldots, n\}$ we will denote by x^N the vector whose components are x_j, for $j \in N$, and by Q_N the matrix of the j^{th} rows and columns of Q, for $j \in N$.

We first prove the following:

Lemma. If y is a vector of random variables, with mean μ and covariance matrix (A), and Q is a positive definite matrix, then the value of a which minimizes

$$\mathcal{E}(y - a) \quad Q(y - a)'$$

is

$$\hat{a} = \mu$$

proof:

$$\mathcal{E}(y - a) \quad Q(y - a)'$$

$$= \mathcal{E}[(y - \mu) \quad Q(y - \mu)' + (a - \mu) \quad Q(a - \mu)' - 2(y - \mu) \quad Q(a - \mu)']$$

$$= \mathcal{E}(y - \mu) \quad Q(y - \mu)' + (a - \mu) \quad Q(a - \mu)'$$

and the second term is minimized only if $a = \mu$. QED

Suppose ω is fixed, and let I be any subset of S. It follows from the lemma that given the values of x_i, $i \in I$, and given that $\omega_i(x_i) = 1$, $i \in I$; $\omega_j(x_j) = 0$, $j \notin I$, the best actions are given by

$$(4) \quad \left\{ \begin{array}{ll} a_i = x_i & , \ i \in I \\ a_j = \hat{a}_j = \mathcal{E}\left\{x_j \mid x_j(\omega_j) = 0\right\} & , \ j \in S-I \\ a_j = \mu_j & , \ j \notin S. \end{array} \right.$$

Recalling that the x_i are independent, if we substitute (4) in (1) and take the expected value, we get for the expected gross payoff:

$$(5) \qquad \mathcal{E}u(\alpha(x), x) =$$

$$- \sum_{I \subset S} \int \prod_{i \in I} \omega_i(x_i) \prod_{j \notin I} (1 - \omega_j(x_j)) \sum_{j \notin I} q_{jj}(x_j - \hat{a}_j)^2 \prod_{i=1}^{n} d\Phi_i(x_i)$$

$$- \sum_{j \notin S} q_{jj} \sigma_j^2$$

$$= -\mathcal{L}(\omega) - \sum_{j \in S} q_{jj} \sigma_j^2$$

The first term on the right of (5) can be simplified by collecting coefficients of each of the terms $(x_j - \hat{a}_j)^2$, to give

$$(6) \qquad \mathcal{L}(\omega) = \sum_{j = \in S} q_{jj} \quad (1 - \omega_j(x_j)) \; (x_j - \hat{a}_j)^2 \; d\Phi_j$$

On the cost side, it is easy to see that the expected operation cost $C(\omega)$ for a given ω equals c times the sum of the probabilities of communicating each of the x_i, i.e.

$$(7) \qquad C(\omega) = c \sum_{i \in S} \int \omega_i(x_i) \; d\Phi_i(x_i).$$

The net expected payoff, not counting the structure cost, is:

$$-\mathcal{L}(\omega) - \sum_{j \notin S} q_{jj} \sigma_j^2 - C(\omega)$$

We now want to minimize this with respect to ω. Combining (6) and (7):

$$(8) \qquad \mathcal{L}(\omega) + C(\omega) = \sum_{j \in S} \left\{ c + \int (1 - \omega_j) \left[q_{jj}(x_j - a_j)^2 - c \right] d\Phi_j \right\}$$

To find the best ω, each term in the above sum can be minimized separately; i.e., we want to minimize expressions of the form

(12) $$\mathcal{L}_j(\omega_j;\ a_j) = \int [1 - \omega_j(x_j)]\ [q_{jj}(x_j - a_j)^2 - c]\ d\Phi_j(x_j)$$

For any fixed a_j the integrand is a linear function of ω_j, thus the best ω_j is given by

(13) $$\omega_j(x_j) = {1 \atop 0} \text{ according as } q_{jj}(x_j - a_j)^2 \text{ is } {> \atop \leq} c$$

$$\text{or as } (x_j - a_j) {> \atop \leq} \sqrt{\frac{c}{q_{jj}}}$$

which is equation (3a).

Hence (12) can be rewritten

$$(14) \quad \min_{\omega_j} \mathcal{L}_j(\omega_j, a_j) = \int_{a_j - \sqrt{\frac{c}{q_{jj}}}}^{a_j + \sqrt{\frac{c}{q_{jj}}}} [q_{jj}(x_j - a_j)^2 - c] \, d\Phi_j(x_j)$$

But condition (5) now becomes

$$\int_{\hat{a}_j - \sqrt{\frac{c}{q_{jj}}}}^{\hat{a}_j + \sqrt{\frac{c}{q_{jj}}}} (x_j - \hat{a}_j) \, d\Phi_j(x_j) = 0$$

or

$$(15) \quad \int_{-\sqrt{\frac{c}{q_{jj}}}}^{\sqrt{\frac{c}{q_{jj}}}} t \, d\Phi_j(t + \hat{a}_j) = 0$$

which is equation (3b).

There may be some question about the existence or uniqueness of the solution $\hat{a}_j$ of (15), but in most cases this will cause no trouble. For example, if Φ_j is absolutely continuous and unimodal, then $\hat{a}_j = \mu_j$. QED

2.1.2 The result of the previous theorem is especially simple because it tells us that the added gross advantage of including a new variable x_j in a given structure S depends only upon the properties of that variable. Thus if we denote by S_o the empty set, and let

$W(S) = V(S) + c(S)$, then

$$(16) \quad W(S_o) = -\sum_{j=1}^{n} q_{jj} \sigma_j^2$$

(17) $$W(S) - W(S_o) = \sum_{i \notin S} W_i$$

where $W_i = q_{ii}\,\sigma_i^2 - \int \min[q_{ii}(x_i - \hat{a}_i)^2,\ c]\ d\Phi_i$.

It is also interesting to note that the expected payoff depends on Q only through its diagonal entries. Both these properties of the solution are, of course, due to the independence of the x_i.

If the operation cost c is zero, then $\hat{\omega}_i(x_i) = 1$ for all x_i. In that case $W_i = q_{ii}\,\sigma_i^2$ and we have

<u>Corollary</u>. If $c = 0$ then for any S

$$V(S) = - \sum_{j \notin S} q_{jj}\,\sigma_j^2 - c(S)$$

Thus the gross advantage of including a variable in S is equal to the variance of that variable weighted by its "importance" in the payoff function.

2.2. <u>Dependent, Normally distributed x_i:</u>

The pure structure problem for case 2, under the assumption that the x_i are normally distributed with means μ_i and covariance matrix $((\sigma_{ij}))$ is easily solved with the aid of the lemma used in section 2.1.1.

Let T be the matrix of the quadratic form of the density function of x, i.e. T is the inverse of the matrix of variances and covariances. The main result is:

<u>Theorem</u> Under the assumptions of this section, for any structure S, the optimal rule of action $\hat{\alpha}$ is given by

(18) $$\hat{\alpha}_i(x^S) = \begin{cases} x_i, & i \in S \\ \mathcal{E}(x_j \mid x^S), & j \notin S. \end{cases}$$

and the expected net payoff is

(19) $V(S) = -tr[Q_{\bar{S}} (T_{\bar{S}})^{-1}] - c(S)$

where $\bar{S} = M - S$

proof. Equation (18) follows immediately from the above mentioned lemma, and (19) follows from substituting (18) in the expected payoff function, and using the familiar facts about conditional variances and covariances of a multinormal distribution.

$(T_{\bar{S}})^{-1}$ is in fact the covariance matrix of the conditional distribution of $x^{\bar{S}}$ given x^{S}. $tr[Q(T_{\bar{S}})^{-1}]$ is nothing more than the sum of these conditional variances and covariances, weighted by the corresponding q_{ij}. $(i, j \notin S)$. $\mathcal{E}(x_j \mid x^S)$ is, of course, a linear function of x^S.

3. Case 2. Choice Between Two Actions.

Introduction.

3.1 Recall that in case 2, the action a can take on only two values, 1 or 0, say, and the net payoff is of the form

(2) $$u(a, x) = \sum_{i=1}^{n} w_i(x_i)$$

Since we can always make the transformation $w_i = w_i(x_i)$ there is no loss of generality in assuming the payoff to be of the form

(2') $$u(a, w) = a \sum_{i=1}^{n} w_i$$

In interpreting results, however, it should be remembered that any properties of the distributions ψ_i of w_i stem from the properties of both the distribution of x and the functions w_i.

This case has not been as completely worked out as the first even under the assumption of independent w_i. We first discuss the pure structure problem for independent w_i and for normally distributed dependent

w_i. Then a theorem concerning the operation problem for $n = 2$ presented, and finally an iterative scheme for solving operation problems is illustrated by a numerical example.

3.2 The Pure Structure Problem: Independent Observables

For any given structure S, and any given value of w^S, the expected gross payoff is

$$\int a\left(\sum_{i \in M} w_i\right) \prod_{j \in M-S} d\,\psi_j(w_j)$$
$$= a\, z_S(w^S)$$

where

$$(20) \qquad z_S(w^S) = \sum_{i \in S} w_i + \sum_{j \in M-S} \mathcal{E}(w_j)$$

Hence the best rule of action given S, is

$$(21) \qquad \hat{\alpha}(w^S) = \begin{smallmatrix}1\\0\end{smallmatrix} \text{ as } z_S(w^S) \gtrless 0$$

and the best expected gross payoff, given S, is

$$(22) \qquad U(S) = \int_{z_S > 0} z_S \, d\Upsilon_S(z_S)$$

where z_S is the random variable defined by $z_S = z_S(\bar{w}^S)$ and Υ_S is its distribution.

Thus the pure structure problem reduces to the study of the distribution of z_S. This much can be said right off: If $\bar{\bar{w}} = \sum_{i \in M} \mathcal{E}(w_i)$ and $\sigma_i^2 = Var(w_i)$ then

$$(23) \qquad \mathcal{E}(z_S) = \bar{\bar{w}} \quad , \text{ for all } S$$

$$s_S \equiv Var(z_S) = \sum_{i \in S} \sigma_i^2$$

If the number of elements of S is "large," the central Limit Theorem will, in general, apply so that $\dfrac{z_S - \bar{\bar{w}}}{s_S}$ will be approximately normally distributed with mean zero and variance 1. If γ and Γ denote the normal density and cumulative distribution functions, respectively, then the gross payoff will approximately be:

$$U(S) \approx \int_{s_S y + \bar{\bar{w}} > 0} (s_S y + \bar{\bar{w}}) \ \gamma(y) \ dy$$

or

$$(24) \quad U(S) \approx s_S \gamma\left(\frac{\bar{\bar{w}}}{s_S}\right) + \bar{\bar{w}} \Gamma\left(\frac{\bar{\bar{w}}}{s_S}\right) = f(s_S, \bar{\bar{w}})$$

Differentiating, we observe that

$$(25) \quad \frac{\partial f}{\partial s_S} = \gamma\left(\frac{\bar{\bar{w}}}{s_S}\right) > 0, \quad \frac{\partial f}{\partial \bar{\bar{w}}} = \Gamma\left(\frac{\bar{\bar{w}}}{s_S}\right) > 0$$

Thus $U(S)$ is (approximately) an increasing function of s_S and of $\bar{\bar{w}}$. This indicates that the larger the variance of a variable w_i, the more advantageous it is to include i in S (cf. (23)), i.e. it is possible to order the variables w_i by their variances so that (from the point of view of <u>gross</u> payoff) the best set S with one element is $\{1\}$, the best set with two elements is $\{1, 2\}$, etc. We shall see that this may not be possible for dependent variables.

<u>3.3</u> <u>The Pure Structure Problem: Normally Distributed Observables</u>

Suppose the w_i have joint normal distribution with means $\bar{\bar{w}}_i$ and matrix $((\sigma_{ij}))$. The analysis of section 3.2 also applies here (with very slight modification) to give the gross payoff as in equation (24) (exactly) except that z_S must now be defined by

$$z_S(w^S) = \sum_{i \in S} w_i + \sum_{j \in M-S} \mathcal{E}(w_j \mid w^S)$$

and s_S, the variance of z_S is no longer simply $\sum_{i \in S} \text{Var}(w_i)$. In this case the value of s_S is given by:

Lemma:

$$(26) \qquad Var(z_S) = \sum_{i,j \in M} \sigma_{ij} - \sum_{i,j \in M-S} \sigma_{ij}(S)$$

where $\sigma_{ij}(S)$ is the conditional covariance of w_i and w_j, given w^S

Proof. Since S is fixed we write $\sigma_{ij}(S) = \bar{\sigma}_{ij}$

Let $((\sigma^{ij})) = ((\sigma_{ij}))^{-1}$, and let Q be the matrix formed from the i^{th} rows and columns of $((\sigma^{ij}))$, for $i \in M-S$. Then it is well known that

$$(26a) \qquad ((\bar{\sigma}_{ij})) = Q^{-1} \qquad \text{(cf. [4], p. 181)}$$

The conditional expections $\mathcal{E}(w_j \mid w^S)$ are given by:

$$\mathcal{E}(w_j \mid w^S) = -\sum_{i \in S} d_{ij} w_i \qquad \text{(cf. [4], p. 183)}$$

$$(27) \qquad d_{ij} \equiv \sum_{k \in \bar{S}} \bar{\sigma}_{jk} \sigma^{ki} \qquad , \bar{S} = M-S$$

$$z_S = \sum_{i \in S} (1 - \sum_{j \in \bar{S}} d_{ij}) W_i$$

$$(28) \qquad z_S = \sum_{i \in S} (1 - b_i) W_i \ , \quad b_i \equiv \sum_{j \in \bar{S}} d_{ij}$$

$$Var(z_S) = \sum_{i,k \in S} (1 - b_i)(1 - b_k) \sigma_{ik}$$

$$(28a) \qquad = \sum_{i,k \in S} \sigma_{ik} - 2 \sum_{i,k \in S} \sigma_{ik} b_i + \sum_{i,k \in S} b_i b_k \sigma_{ik}$$

From (27) and (28) we have

$$b_i = \sum_{j \in \bar{S}} \sum_{\ell \in \bar{S}} \bar{\sigma}_{j\ell} \sigma^{\ell i}$$

$$\sum_{i \in S} \sigma_{ik} \beta_i = \sum_{i \in S} \sum_{j,\ell \in \bar{S}} \sigma_{ik} \bar{\sigma}_{j\ell} \sigma^{\ell i}$$

(29) $$= \sum_{j,\chi \in \bar{S}} \bar{\sigma}_{j\chi} \sum_{i \in S} \sigma_{ik} \sigma^{\chi i}$$

Applying the definition of $((\sigma^{ij}))$:

$$\sum_{i \in M} \sigma_{ik} \sigma^{\chi i} = \delta_{k\chi} \begin{cases} 1 & \text{if } k = \chi \\ 0 & \text{if } k \neq \chi \end{cases}$$

also $$\sum_{i \in M} \sigma_{ik} \sigma^{\chi i} = \sum_{i \in S} \sigma_{ik} \sigma^{\chi i} + \sum_{i \in \bar{S}} \sigma_{ik} \sigma^{\chi i}$$

Hence $$\sum_{i \in S} \sigma_{ik} \sigma^{\chi i} = \delta_{k\chi} - \sum_{i \in \bar{S}} \sigma_{ik} \sigma^{\chi i}$$

Substituting in (29)

$$\sum_{i \in S} \sigma_{ik} b_i = \sum_{j,\chi \in \bar{S}} \sigma_{j\chi}(\delta_{k\chi} - \sum_{i \in \bar{S}} \sigma_{ik} \sigma^{\chi i})$$

For $k \in S$ and $\chi \in \bar{S}$, $\delta_{k\chi} = 0$. Hence

$$\sum_{i \in S} \sigma_{ik} b_i = - \sum_{i,j,\chi \in \bar{S}} \bar{\sigma}_{j\chi} \sigma_{ik} \sigma^{\chi i}$$

$$= - \sum_{i \in \bar{S}} \sigma_{ik} \sum_{j \in \bar{S}} \sum_{\chi \in \bar{S}} \sigma_{j\chi} \sigma^{\chi i}$$

$$= - \sum_{i \in \bar{S}} \sigma_{ik} \sum_{j \in \bar{S}} \delta_{ji} \qquad \text{(by (26))}$$

(30) $$\sum_{i \in S} a_{ik} b_i = - \sum_{i \in \bar{S}} \sigma_{ik}$$

(31) $$\sum_{i,k \in S} \sigma_{ik} b_i = - \sum_{k \in S} \sum_{j \in \bar{S}} \sigma_{jk}$$

Also from (30):

$$\sum_{i,k \in S} \sigma_{ik} b_i b_k = - \sum_{k \in S} \sum_{j \in \bar{S}} \sigma_{jk} b_k$$

$$= - \sum_{k \in S} \sum_{j,n,m \in \bar{S}} \sigma_{jk} \bar{\sigma}_{nm} \sigma^{mk}$$

$$= - \sum_{j,n,m \in \bar{S}} \bar{\sigma}_{nm} \sum_{k \in S} \sigma_{jk} \sigma^{mk}$$

$$= -\sum_{j,n,m\in\bar{S}} \bar{\sigma}_{nm} \left(\delta_{jm} - \sum_{k\in\bar{S}} \sigma_{jk} \sigma^{mk} \right)$$

$$= -\sum_{j,n\in\bar{S}} \bar{\sigma}_{nj} + \sum_{j,n,m,k\in\bar{S}} \sigma_{jk} \bar{\sigma}_{nm} \sigma^{mk}$$

$$= -\sum_{j,n\in\bar{S}} \bar{\sigma}_{nj} + \sum_{j,n,k\in\bar{S}} \sigma_{jk} \delta_{nk}$$

$$(32) \quad \sum_{i,k\in S} \sigma_{ik} b_i b_k = -\sum_{j,n\in\bar{S}} \bar{\sigma}_{nj} + \sum_{j,n\in\bar{S}} \sigma_{jn}$$

Substitute (31) and (32) in (28a):

$$\mathrm{Var}(z_S) = \sum_{i,k\in S} \sigma_{ik} + 2\sum_{k\in S} \sum_{j\in\bar{S}} \sigma_{jk} + \sum_{j,n\in\bar{S}} \sigma_{jn} - \sum_{j,n\in\bar{S}} \bar{\sigma}_{jn}$$

Combining the first 3 terms gives equation (26) proving the lemma.

Looking at equation (26), we see that it is the term $-\sum_{i,j\in\bar{S}} \sigma_{ij}(S)$ which varies with S. Equation (26a) tells us that there is no hope, in general, of ordering the variables w_i so that (from the point of view of gross payoff) the best set S with one element is $\{1\}$, the best set with two elements is $\{1, 2\}$, etc.

3.4. An Operation Problem With 2 Observers

3.4.1 **Theorem:** If the W_1 and W_2 are independent and $S = \{1, 2\}$, then the best operation ω has the property that, for each i, the set of w_i such that $\omega_i(w_i) = 0$ is an interval (possibly infinite or degenerate).

The proof of this theorem is given in [5]. In addition, a necessary condition is given which in many cases would be sufficient to determine the optimal ω. It is conjectured that the theorem holds for any number of observers, but this has not yet been proven.

3.4.2. Numerical Solution by Iteration.

Although the above theorem reduces the operation problem to one of maximizing a function of only four variables, the actual computation of the best operation is usually very messy. The following iterative scheme suggests itself, both as a method of computation and as a possible description of the way in which real organizations work out their rules of operation.

Suppose ω_1 is fixed, say in the form ω_1^o. Let ω_2^1 be the best form of ω_2, given $\omega_1 = \omega_1^o$. Let ω_1^1 be the best form of ω_1, given $\omega_2 = \omega_2^1$, etc. If the expected payoff as a function of ω is sufficiently "smooth," then the sequences ω_1^n and ω_2^n will converge to some operations $\hat{\omega}_1$ and $\hat{\omega}_2$, which will determine a local maximum of the expected payoff function.

This method is being applied to the following example, although thus far only a few results have been obtained:[1]

(i) $u(a, x) = a(\xi_1 x_1 + \xi_2 x_2)$

(ii) x_1 and x_2 are independent and normally distributed with means zero and variances one.

(iii) c is the cost of any one communication.

The results of the computations thus far completed are summarized in the following table. ω is described by:

$$\omega_i(x_i) = \begin{cases} 0 & \text{if } r_i \leq x_i \leq s_i \\ 1 & \text{otherwise} \end{cases}$$

---

[1] The computations for this section were carried out by Mr. Toharu Miyashita, who was also very helpful in suggesting improvements in the computational scheme and in working out many of the details.

The action taken if neither x_i is reported (i.e. if $\omega_1(x_1) = \omega_2(x_2) = 0$) is denoted by q_o. For all the solutions given in the table, $q_o = 1$. Another set solutions (r_i^1, s_i^1) can be obtained by letting $q_o = 0$, and setting $r_i^1 = -s_j$ and $s_i^1 = -r_j$. This is due to the symmetry of the example; in general one of the values of q_o will be the best.

Table: Numerical solution of 1 actor - 2 observer operation problem: $\xi_2 = 1$, $q_o = 1$

ξ_1 \ c		0.1		1.0		2.0	
		1	2	1	2	1	2
1	r	-.483	-.483	-1.297	-1.297	-2.08	-2.08
	s	1.302	1.302	12.10	12.10	110.0	110.0
	U	.4480		.116		.015	
2	r	.0625	-1.854				
	s	2.677	.886				
		.7772					

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