COWLES COMMISSION DISCUSSION PAPER: ECONOMICS NO. 2076



# Structural and Operational Communication Problems in Teams, I.

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May 13, 1953

Contents



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1. Research undertaken by the Cowles Commission for Research in Economics under Contract Nonr-358(01), NR 047-006 with the Office of Naval Research.

1. Problem Formulation.

1.1. Introduction. In a previous paper [1] the concept of a team was discussed as part of a general approach to a rational theory of organizations. In this paper we shall formulate, in a precise but general fashion, an important aspect of the team problem, and investigate in detail some examples.

A second paper [2] will deal with two classes of team problems which are of the general type described here.

1.2. The General Problem. Consider a team problem defined as follows: Assume an equal number m of action variables $a_i$ and random observable variables $x_i$.

Let the $x_i$ have a joint distribution $\Phi(x_1, \ldots, x_m)$. Denoting by a and x the m-tuples of actions and observable random variables, respectively, the gross payoff to the team is some (real valued) function $u(a,x)$ of a and x. The expected gross payoff, $\int_x u(a,x)d\Phi(x)$ will be sometimes denoted by U.

Obviously the expected gross payoff is the largest when the action a of the team as a whole is based upon knowledge of the entire m-tuple x. At the other extreme is the case in which a is determined independently of the value of x (but of course in full knowledge of the distribution $\Phi$). This suggests that for intermediate cases we have to define, for each i, the "vector" $x^{(i)}$ upon which the action $a_i$ is based.[1] The values of the components of $x^{(i)}$ are communicated to the i-th member by his fellow members or are observed by himself. $x^{(i)}$ is his "state of information."

1. The word "vector" here is used loosely, since we have not yet specified the nature of the $x_i$: for example, $x_i$ may or may not be a real number.

Let M be the set of integers from 1 to m, and for every i let $S_i$ be a subset of M. Let $x^{(i)} = \{x_j\}_{j \in S_i}$ be the vector of all $x_j$ such that j is in $S_i$. Then for any given m-tuple $S = \{S_i\}_{i \in M}$ we conceive of the action $a_i$ being based upon the vector $x^{(i)}$. S describes the communication network of the team and will be called its structure.

We must also consider the more general possibility that $a_i$ is sometimes based only upon a part of the vector $x^{(i)}$, depending upon the actual value of $x^{(i)}$; for, depending on $x^{(i)}$, messages may or may not be sent to the i-th member. Thus let $y^{(i)}$ be a vector with the same number of components as $x^{(i)}$, the value of $y_i$ being given by a function $\omega^{(i)}$, such that for any component $\omega_j^{(i)}$ of $\omega^{(i)}$

$$\omega_j^{(i)}(x_j) = \begin{cases} x_j \text{ or} \\ \ominus \end{cases}$$

where the symbol $\ominus$ is to be translated "no message about $x_j$." The vector $\omega = \{\omega^{(i)}\}_{i \in M}$ of functions $\omega^{(i)}$ is the rule of how to operate the communication network of the team and will be called rule of operation. If $\omega$ is such that the i-th member always receives messages about all components of $x^{(i)}$, then $x^{(i)} = y^{(i)}$ represents his "state of information"; otherwise $y^{(i)}$ represents his "state of information," and $x^{(i)}$ represents his "maximum state of information."

The notion that the actions are based on information about the observables is formalized by saying that each $a_i$ is determined by a function $\alpha_i$ whose argument is $y^{(i)}$, i.e.,

$$a_i = \alpha_i(y^{(i)}) .$$

The vector $\alpha = \{\alpha_i\}_{i \in M}$ is called the rule of action. Any combination $(S, \omega, \alpha)$ is a constitution of the team.

Finally, we consider the communication costs associated with a particular constitution. We make in most of this paper (with exceptions stated in Examples A and B: Sections 2.2 and 3.2) the simplifying assumption that the costs are additive, in the following sense: To every structure S is attached

(a) a fixed cost $c(S)$; this is the cost of maintaining and amortizing communication facilities, per period of time needed to make one action; $c(S)$ does not depend on whether a message is actually sent;

(b) the operating cost, $C(\omega)$; this is a random variable defined by

$$C(\omega) = \sum_{i \in M} \sum_{j \in S_i} C_{ij}, \text{ where}$$

$$C_{ij} = \begin{Bmatrix} c_{ij} \\ 0 \end{Bmatrix} \text{ according as } \omega_j^{(i)}(x_i) = \left\{ x_j \right.$$

(In all the cases we consider $c_{ij} = c$ for all i, j.)

The net payoff to the team is

$$v = u(\alpha(x), x) - c(S) - C(\omega),$$

and the expected net payoff will be denoted by V. We can now define the problem of the team: choose that constitution $(S, \omega, \alpha)$ which maximizes the expected net payoff V, subject, possibly, to certain constraints on S, $\omega$ and $\alpha$.

Here is a summary of the notation introduced above:

## Notation Summary

$M = \{1, \ldots, m\}$

$a = (a_1, \ldots, a_m)$, actions

$x = (x_1, \ldots, x_m)$, observable random variables, with distribution $\Phi(x)$

$u(a, x)$, gross payoff $\qquad U = \int_X u \, d\,\Phi(x)$, expected net payoff

$S = \{S_i\}_{i \in M}$, $S_i \subseteq M$, the structure

$x^{(i)} = \{x_j\}_{j \in S_i}$

$y^{(i)} = \omega^{(i)}(x^{(i)})$, the information about $x^{(i)}$ on which $a_i$ is based, where

$\omega^{(i)} = \{\omega_j^{(i)}\}_{j \in S_i}$, $\qquad \omega_j^{(i)}(x_j) = \begin{cases} x_j \text{ or} \\ \theta \end{cases}$

$\omega = \{\omega^{(i)}\}_{i \in M}$, the rule of operation

$a_i = \alpha_i(y^{(i)})$, $\alpha = \{\alpha_i\}$, the rule of action

$c(S)$, the cost of the structure S

$C(\omega) = \sum_{i \in M} \sum_{j \in S_i} C_{ij}$, the cost of operation, where

$$C_{ij} = \begin{cases} c_{ij} \\ \theta \end{cases} \text{ according as } \omega_j^{(i)}(x_j) = \begin{cases} x_j \\ \theta \end{cases}$$

$v = u(\alpha(x), x) - c(S) - C(\omega)$, net payoff

$V = \int_X v \, d\,\Phi(x)$, expected net payoff.

1.3. Special Cases of the General Problem.

1.3.1. Pure Structural Problems. In many instances the operation cost $C(\omega)$ may be negligible compared with the differences between the costs $c(S)$ attached to the various structures $S$ under consideration. If for every $S$ and every $\omega$ $C(\omega)$ is actually zero (with probability one), then an optimal rule of operation is "always communicate":

$$\omega_j^{(i)}(x_j) \equiv x_j \quad , \text{ all } j \in S_i \; , \; i \in M \; .$$

In this case it remains to find the best $S$ and $\alpha$, and this will be called a pure structural problem.

1.3.2. Pure Operational Problems. On the other hand the situation may be such that the structure is fixed, leaving the best $\omega$ and $\alpha$ to be found. This will be called a pure operational problem.[1]

1.3.3. Decentralization. A special case of some interest arises when we consider a team consisting of m members, the i-th member performing an action $a_i$ and observing $x_i$, and where each member belongs to one of the disjoint "sub-teams," such that each $a_i$ is based upon all the $x_i$ observable by the members of his sub-team. This restricts the structure $S$ to be such that, for every i and j in $M$, either

$$S_i \cap S_j = 0, \quad \text{or}$$

$$S_i = S_i \; ,$$

and also $i \in S_i$, for all i.

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1. In [1], all problems with the exception of those of Section 3 were pure operational problems; Section 3 dealt with a pure structural problem.

The structures "no-communication" and "two-way communication" of Example B will deal with the case $m=2$. A more general decentralization problem is treated in [2].

1.3.4. One Actor-n Observers. Another important special case arises when the actions of all but one of the members are fixed, say:

$$a_i = \text{constant}, \quad i \geq 2.$$

(This may also be interpreted as meaning that m-1 members perform no "actions" at all, but can only observe.)

In this case, the only part of the structure that need be considered is the set $S_1$. We may think of those j not in $S_1$ as representing observers who were dismissed.

Example A illustrates this case with $n=1$. A more general discussion is given in [2].

2. Example A: One observable, one action variable.

2.1. Pure Structural Problem.

We first consider the case when there is one observable and one action, and denote them by x and a respectively. In the pure structural problem we have to choose between the following two structures:

$S^1$: x is observed

$S^0$: x is not observed.

In both cases the distribution $\bar{\Phi}(x)$ is known. The gross payoff is in both cases a random quantity $u(a,x)$, where $a = \alpha(x)$, $\alpha$ being the rule of action. The gross expected payoff reaches its maximum value $U=\dot{U}$ when the best rule of action is used, $\alpha = \hat{\alpha}$:

$$\hat{U} = \max_{\alpha} E\, u(\alpha(x),x) \quad .$$

If the structure is $S^o$, the function $\alpha(x)$ degenerates into a constant, a, independent of x.

$S^o$ can be interpreted as a team consisting of a single man, an "actor." $S^1$ can be interpreted in two manners:

Interpretation (I): $S^1$ is a team consisting of a single man ("observer-actor") who is provided with more or less costly physical facilities for observations, or uses a non-negligible part of his time for observations.

Interpretation (II): $S^1$ is a two-men team — an actor and an observer — with more or less costly physical facilities for observation and communication. In the pure structural problem the cost of using a communication line, once such a line is established (or once a fixed part of the team's time is set aside for communications), is negligible, so that the observed value of x is always communicated by the observer to the actor, regardless of the observed value of x (e.g., regardless of whether there is or there is no "emergency").

To fix ideas, suppose the team is speculating in a commodity. If x is its anticipated price change, and a is the amount of commodity sold (with $a > 0$) or bought (with $a < 0$), the payoff is

$$u(a, x) = a\,x.$$

As to the distribution of x, we shall assume it uniform, with mean $\mu$ and range $2\rho$.

We shall assume a to be bounded from above and below; and choose the scale and origin of a so as to make these bounds +1 and -1, respectively. As will be verified easily, this will not change our solution essentially. In fact, our results will apply to any linear payoff function $u = a \cdot f(x) + g(x)$, provided f(x) is distributed uniformly.

If the structure is $S^o$, the best action $a=\hat{a}$ is defined by

$$\hat{U}=\hat{U}(S^o)=\max_a E\, a\, x=\hat{a}\,\mu\,, \quad -1 \le \hat{\alpha} \le +1$$

Hence $\hat{a}=+1$ if $\mu \ge 0$; and $\hat{a}=-1$ if $\mu \le 0$. And since the cost of observation facilities (or the time devoted to observations) is zero, the maximum net payoff is

$$(2.1) \qquad \hat{V}(S^o) = \hat{U}(S^o) = |\mu|.$$

In $S^1$, on the other hand, action does depend on $x$. The best rule of action $\alpha=\hat{\alpha}$ maximizes $U(S^1)$:

$$\hat{U}=\hat{U}(S^1) = \max_\alpha E\, \alpha(x)x = E\, \hat{\alpha}(x)x\,, \quad -1 \le \alpha(x) \le 1.$$

Hence the rule $\hat{\alpha}$ is

$$\hat{\alpha}(x) = \begin{cases} -1\,, & x \le 0 \\ +1\,, & x \ge 0\,. \end{cases}$$

Since $\mu - \rho \geqslant 0$ implies that always $x \geqslant 0$, and $\mu + \rho \leqslant 0$ implies that always $x \le 0$, we can write:

$$\hat{U}(S^1) = |\mu| \quad , \qquad |\mu| \geqslant \rho.$$

If, on the other hand, $-\rho \le \mu \le \rho$, we have

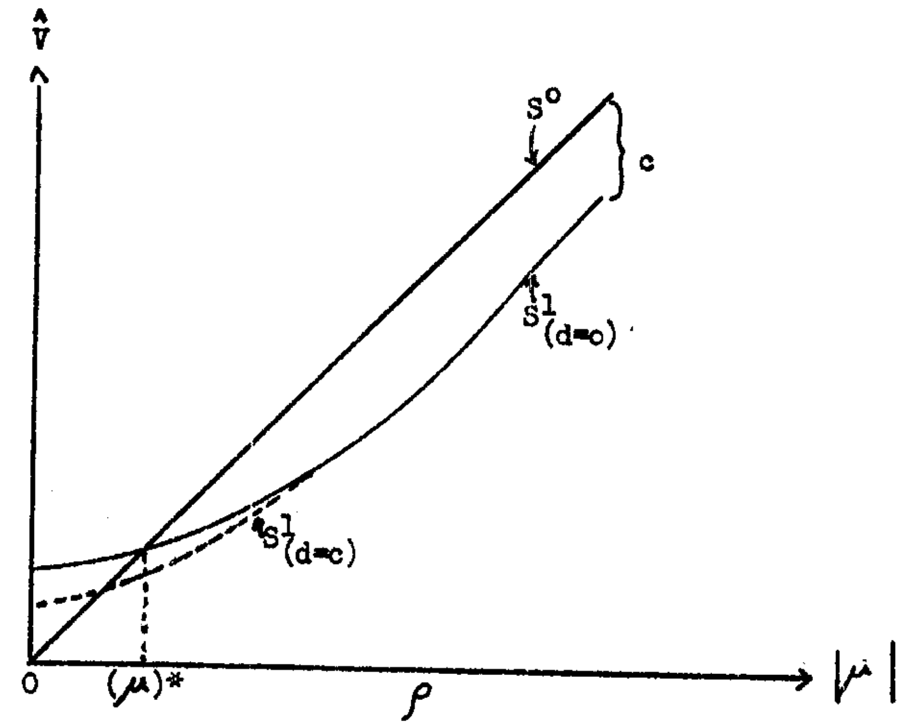
$$\hat{U}(S^1) = \frac{1}{2\rho}\left(\int_0^{\mu+\rho} x\,dx - \int_{\mu-\rho}^{0} x\,dx\right) = \frac{\mu^2+\rho^2}{2\rho} \quad , \quad |\mu| \le \rho\ .$$

We have to subtract from $U(S^1)$ the fixed cost $c(S^1)=c$. It is the cost, per time-period required for an action, of amortizing and maintaining observation and communication facilities. In the two-men team interpretation of $S^1$, stated as Interpretation (II) above, the amount $c$ includes also the maintenance of the observer. The maximum net payoff is

$$\hat{V}(S^1) = \hat{U}(S^1) - c =$$

$$(2.2) \qquad \begin{cases} |\mu| - c & \text{if } |\mu| \geqslant \rho \\ \dfrac{\mu^2+\rho^2}{2\rho} - c & \text{if } |\mu| \le \rho. \end{cases}$$

Chart 2.I




Net payoffs of structures $S^o$ and $S^1$

See Equations (2.1), (2.2) and (2.4)

On Chart 2.I, $\hat{V}(S^c)$ and $V(S^1)$ have been plotted [1] against $|\mu|$, for given $\rho$ and $c = \rho/4$. The Chart shows the critical value $|\mu|^*$ of the mean $\mu$: $S^1$ is preferred to $S^o$ if $|\mu| < |\mu|^*$. That is, if the distribution of x is known to be strongly biased, it is not worthwhile to learn the values of x.

Alternatively, one may be interested in the critical cost $c^*$, such that $S^1$ is rejected in favor of $S^o$ whenever $c > c^*$. Critical cost $c^*$ is identical with the "value of inquiry," w, discussed in another paper [2] and is

$$c^* = \begin{cases} 0 & \text{if } |\mu| \geq \rho \\ (|\mu| - \rho)^2/2\rho & \text{if } |\mu| \leq \rho . \end{cases}$$

The pure structural problem can be stated thus: in the parameter space $(\mu, \rho, c)$ find the "acceptance region for $S^1$," i.e., the region in which $\hat{V}(S^1) \geq \hat{V}(S^o)$. Because of (2.1), (2.2) this region is given by

$$(2.3) \qquad \rho - \sqrt{2\rho c} \geq |\mu| .$$

On Chart 2.II, the acceptance region for $S^1$ lies above each of the two curves corresponding to two fixed values of c.

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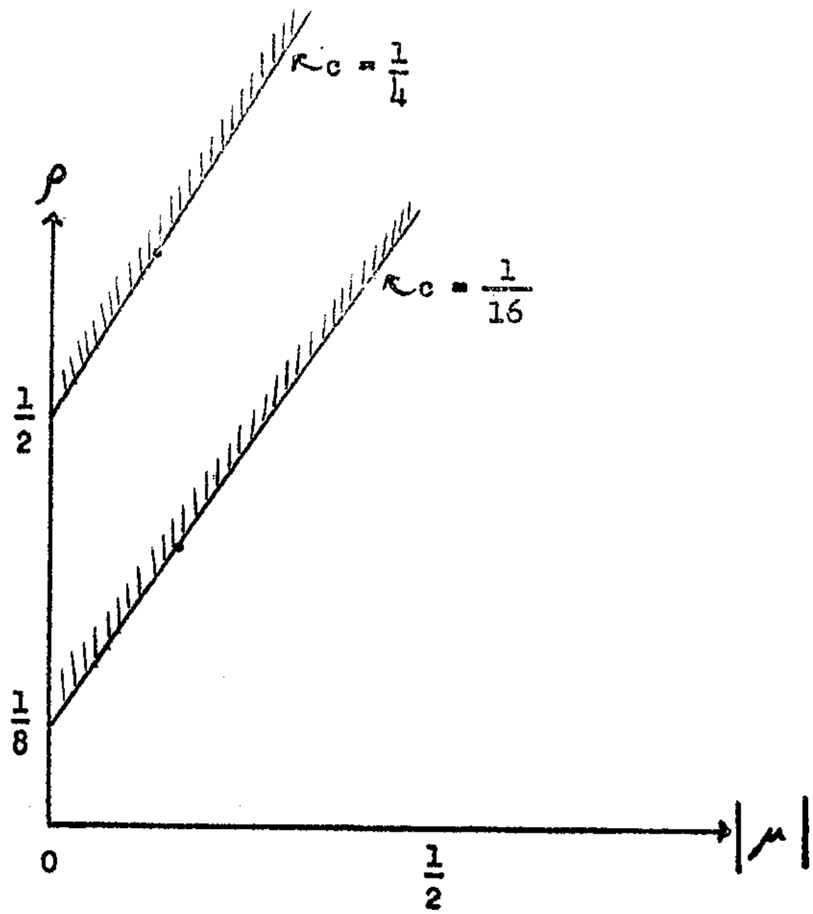
1. On the Chart, two $S^1$ - curves are drawn: $S^1_{(d=o)}$ and $S^1_{(d=c)}$. We discuss here only the former one: d(the cost of operating the communication line) is, so far, assumed zero, since we are concerned with a pure structural problem.

2. Please correct formula (3.4.2) in CCDP Economics 2051.

Chart 2.II

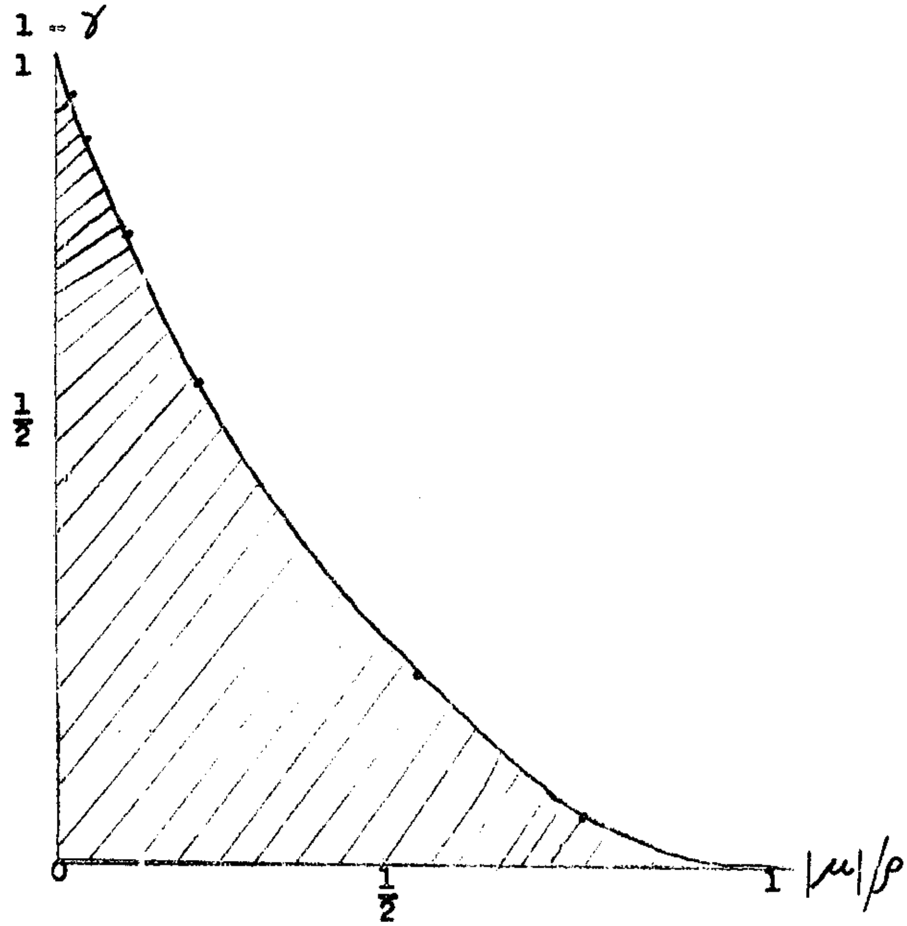



**Regions of acceptance of $S^1$ against $S^0$, depending on distribution of x and on fixed cost c of observation and communication. See equation (2.3).**

This is the shaded region of Chart 2.III. The larger the ratio $\frac{|\mu|}{\rho}$ of the distribution of x the smaller is the proportion $(1-\gamma)$ of time that it is worthwhile to set aside for observation and communication.

Chart 2.III

See Equation (2.3')




Region of acceptance of $S^1$ against $S^0$, depending on distribution of x and on proportion $(1-\lambda)$ of time set aside for observation and communication.

2.3. Pure Operational Problem.

We now proceed to the pure operational problem. For the structure $S^o$ (in which no observations are made) and for the one-man team interpretation (Interpretation (I)) of the structure $S^1$ (in which observations are made but no reports are sent) this problem does not exist. It exists for the two-man team interpretation (Interpretation (II)) of $S^1$. Given the cost of sending a message, one has to find the best operational rule. Denote the message received by an actor from the observer by

$$y = \omega(x) = x \text{ or } \theta$$

where $\theta$ means "no message" and $\omega$ is the operational rule. The cost is

$$D = \begin{matrix} d \\ 0 \end{matrix} \text{ if } \omega(x) = \begin{matrix} x \\ \theta \end{matrix} .$$

(D and d correspond to $C_{ij}$ and $c_{ij}$ of our general notation.) The rule of action $\alpha$ has now as its argument, not x but the message y; and the net expected payoff is

$$V = V(S^1; \omega) = E[\alpha(y) \cdot x - D] - c .$$

We have seen before that in $S^1$ the only desirable actions are $\hat{a} = +1$ and $\hat{a} = -1$, the former when $x > 0$, the latter when $x \leq 0$. There are therefore two possible classes of operational rules for the observer:

$$\Omega^+ : \text{call only if } \hat{a} = +1$$

$$\Omega^- : \text{call only if } \hat{a} = -1 .$$

Consider first a rule $\omega$ in $\Omega^+$. Then

$$\omega(x) = y = x \text{ implies } \hat{a} = \hat{\alpha}(y) = 1,\ \hat{a}\, x = x \quad , D = d$$

$$\omega(x) = y = \theta \text{ implies } \hat{a} = \hat{\alpha}(y) = -1,\ \hat{a}\, x = -x, D = 0$$

Hence, to maximize $V = E(a\,x - D) - c$, one has to let $\omega(x) = x$ and $\alpha(y) = 1$ whenever $x - d \geq -x$, i.e., $x \geq d/2$; and to let $\omega(x) = \theta, \alpha(y) = -1$ whenever $x \leq d/2$. The function $\omega = \hat{\omega}^+$ thus found yields the maximum net payoff

$$\hat{V}^+ = V(S^1, \hat{\omega}^+) =$$

$$= \begin{cases} \mu - d - c & \text{if } \frac{d}{2} \leq \mu - \rho \qquad (1) \\ -\mu - c & \frac{d}{2} \geq \mu + \rho \qquad (2) \\ -\mu - c + (\rho + \mu - \frac{d}{2})^2/2\rho & \text{otherwise.} \qquad (3) \end{cases}$$

Similarly if $\omega$ is in $\Omega^-$, the best rule, $\hat{\omega}^-$, is for the observer to call when $x \leq -\frac{d}{2}$ (in which case the actor makes $x = -1$). The payoffs are

$$\hat{V}^- = V(S^1, \hat{\omega}^-) =$$

$$= \begin{cases} -\mu - d - c & \text{if } \frac{d}{2} \leq -\mu - \rho \qquad (1') \\ \mu - c & \frac{d}{2} \geq -\mu + \rho \qquad (2') \\ \mu - c + (\rho - \mu - \frac{d}{2})^2/2\rho & \text{otherwise.} \qquad (3') \end{cases}$$

We have to choose the best operational rule as $d > 0$ changes its value over the 9 intersections of each of the intervals (1), (2), (3) with each of the intervals (1'), (2'), (3'). We note that

$$\text{in (1)} \quad \mu > \rho > 0; \quad \text{in (1')} \quad \mu < -\rho < 0 \; ;$$

$$\text{in (3)} \quad \mu > -\rho ; \quad \text{in (3')} \quad \mu < \rho .$$

Hence $(1) \cap (1')$, $(1) \cap (3')$, $(1') \cap (3)$ are empty. For the remaining 6 intersections, we have to study the sign of $\Delta = \hat{V}^+ - \hat{V}^-$, and to choose $\hat{V} = \max(\hat{V}^+, \hat{V}^-)$.

$(1) \cap (2')$: $\Delta = -d < 0$ ; $\mu < 0$ ; $\hat{V} = V^- = |\mu| - c$;

$(2) \cap (1')$: $\Delta = d > 0$ ; $\mu > 0$ ; $\hat{V} = V^+ = |\mu| - c$;

$(2) \cap (2')$: $\Delta = -2\mu$ ; $\hat{V} = \max(V^-, V^+) = |\mu| - c$;

$$(3)\cap(2'): \Delta = [-2\rho\mu + (\rho-\mu)^2 - \frac{d^2}{4} - d\,(\rho+\mu-\frac{d}{2})]\,/2\rho < 0,$$

because $\mu > 0$ and $|\rho-\mu| \leq \frac{d}{2} \leq \rho+\mu$; $\hat{v} = v^- = |\mu| - c$;

$$(2)\cap(3'): \Delta = -[2\rho\mu + (\rho+\mu)^2 - \frac{d^2}{4} - d\,(\rho-\mu-d)]\,/2\rho > 0,$$

because $\mu < 0$ and $|\rho+\mu| \leq \frac{d}{2} \leq \rho-\mu$; $\hat{v} = v^+ = |\mu| - c$;

$$(3)\cap(3'): \Delta = -\mu d/\rho; \quad \hat{v} = \max(v^-, v^+) = |\mu| + \frac{(\rho-|\mu|-\frac{d}{2})^2}{2\rho} - c\,.$$

As one would expect, $\omega^+$ (the rule of sending a message whenever x is high) is discarded in cases when $\mu > 0$ (i.e., in cases when x is more often negative than positive), and $\omega^-$ is discarded when $\mu < 0$. Moreover, with the exception of the case $(3)\cap(3')$, the preferred rule will exclude communication altogether and yield the net payoff $\hat{v} = |\mu| - c$. The case $(3)\cap(3')$ is: $\frac{d}{2} \leq \rho - |\mu|$, since d and $\rho$ are non-negative. We sum up the result:

| Conditions on $(d, \rho, \mu)$ | | Best rule of operation | Net payoff $\hat{v}$: |
|---|---|---|---|
| $\frac{d}{2} \leq \rho - \lvert\mu\rvert$, | $\mu < 0$ | $\omega(x) = \begin{cases} x, & x \geq \frac{d}{2} \\ \theta, & x \leq \frac{d}{2} \end{cases}$ | $-c + \lvert\mu\rvert + (\rho - \lvert\mu\rvert - \frac{d}{2})^2/2\rho$ |
| | $\mu > 0$ | $\omega(x) = \begin{cases} x, & x \leq -\frac{d}{2} \\ \theta, & x \geq \frac{d}{2} \end{cases}$ | |
| $\frac{d}{2} > \rho - \lvert\mu\rvert$ | | $\omega(x) = \theta$ | $-c + \lvert\mu\rvert$. |

This completes the pure operational problem. On a Chart which would be analogous to Chart 2.II one might indicate the region in the $(|\mu|, \rho)$-plane, [redacted]ich communications (for x exceeding $\frac{d}{2}$ or falling short of $-\frac{d}{2}$) would[redacted] all justified.

The boundary would this time be a straight line, with slope 1 and intercept $\frac{d}{2}$.

2.4. Complete, or constitutional, problem.

We can now combine the purely structural and the purely operational problem into a general problem of the best constitution $(S, \omega)$, S being defined over the set $(S^o, S^1)$ with the operation cost $d > 0$. The difference $\hat{V}(S^1)-\hat{V}(S^o)$ is

$$\begin{array}{ll} -c + (\rho - |\mu| + \frac{d}{2})^2/2 & \text{if } \frac{d}{2} \leq \rho - |\mu| \\ -c & \text{otherwise.} \end{array}$$

Hence, if condition 2.3 is generalized into

(2.4) $$\rho - \sqrt{2\rho c} - \frac{d}{2} \geq |\mu| ,$$

it becomes the condition for accepting $S^1$, given d and assuming that the best operational rule is chosen. This condition can be visualized on Chart 2.II, if the curves are shifted to the left by the horizontal amount $\frac{d}{2}$.

The effect of introducing the cost (d) of operating the communication channels is also represented on Chart 2.I. In addition to the net payoffs in the case $S^o$ and in the case $S^1$ with $d=0$, it shows the payoff in the case $S^1$ with $d=c$. As one would expect, the new $S^1$ curve intersects the $S^o$ curve at a lower point than the old one.

Example B: Two observables, two action variables.

3.1. Pure Structural Problem.

The model considered will involve two observer-actors. The member i observes the variable $x_i$ and performs the action $a_i$. In addition there may or may not be facilities for communication between the two; or a fixed part of the members' time may or may not be set aside for communication. There are four rival structures (see Chart 3.I):

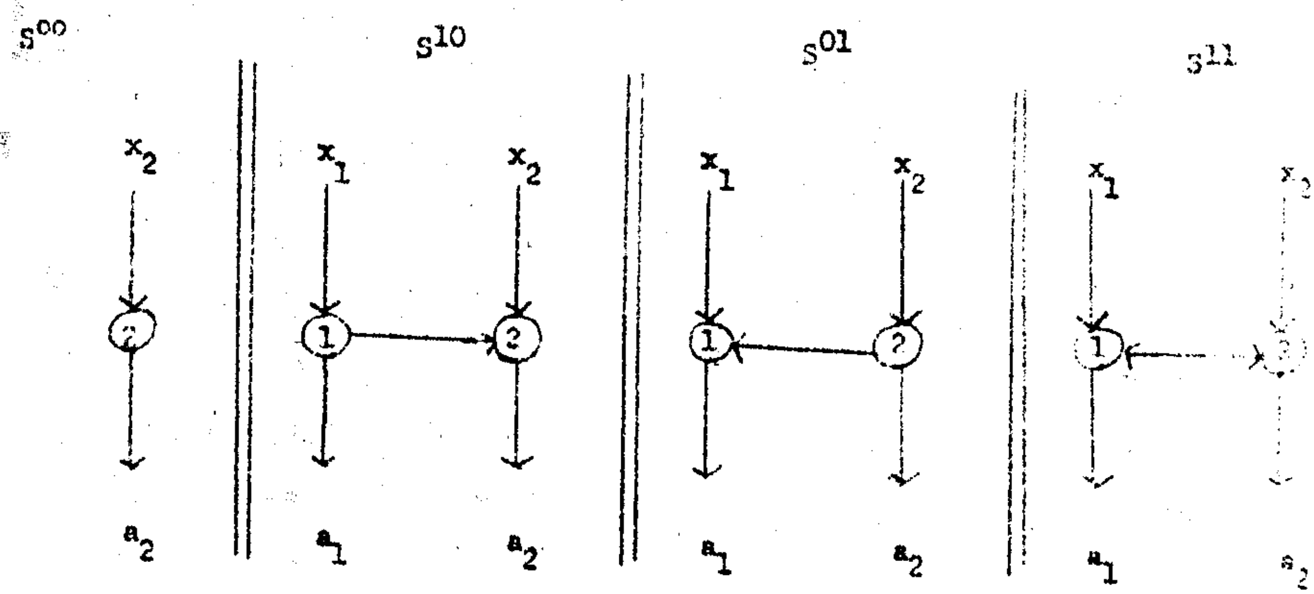
$S^{oo}$: No communication.

$S^{10}$: One-way communication: 1 reports to 2.

$S^{01}$: One-way communication: 2 reports to 1.

$S^{11}$: Two-way communication.

Chart 3.I




Hence, before he acts, each member has at its disposal the following information $x^{(i)}$:

In $S^{oo}$, $x^{(i)} = (x_i)$ , $i = 1, 2$

In $S^{10}$, $x^{(1)} = (x_1)$ , $x^{(2)} = (x_1, x_2)$

In $S^{01}$, $x^{(1)} = (x_1, x_2)$ , $x^{(2)} = x_2$

In $S^{11}$, $x^{(i)} = (x_1, x_2)$ , $i = 1, 2.$

The action $a_i = \alpha_i (x^{(i)})$, where $\alpha_i$ is the action rule. The payoff is $u(a_1, a_2, x_1, x_2)$.

In the pure structural problem, once a communication line is established, no additional cost is attached to operating it. The communication will therefore be always made, regardless of the particular values of $x_1$ or $x_2$ that have to be communicated (e.g., regardless whether there is an "emergency" or not).

Analogous to the Example A, the following gross payoff function will be assumed:

$$u(a_1, a_2, x_1, x_2) = (a_1 + a_2)(x_1 + x_2).$$

For example, $x_1$ may be the future change in the price of the finished product, as anticipated (correctly, we assume) by partner 1, a specialist in this market; similarly, partner 2 specializes in the market of the raw material and anticipates (correctly) a change ($= -x_2$) in the price of raw material. The amounts of raw material are measured in units such that one unit of finished product is made out of one unit of raw material. Thus, if the firm commits itself to sell and buy one unit of finished product and raw material, respectively, it makes a speculative profit $x_1 + x_2$. We assume that, on the basis of his information, each partner can commit the firm to selling the amount $a_i$ of finished product and buying at the same time the same amount of raw material. The total profit from such commitments is $(a_1 + a_2)(x_1 + x_2)$. Our results will apply (apart from trivial modifications) to any u-function linear in $a_1$, $a_2$.

As in Example A, we assume $a_i$ to be bounded from below and above. Because of the linearity of the payoff function with respect to $a_1$, $a_2$ we can choose the bounds arbitrarily. It will suit our economic illustration if we make $0 \leq a_i \leq 1$. Also because of the linearity of the payoff function, the only values of $a_i$ that result in maximum payoff can be 0 or 1. We can therefore confine ourself to

$$\alpha_i(x^{(i)}) = a_i = 0 \text{ or } 1, \quad i=1, 2.$$

We shall assume $x_1$ and $x_2$ to have independent distributions. Specifically, $x_i$ will be distributed uniformly with mean 0 and range $2\rho_i$. We shall assume $\rho_1 > \rho_2$.

Consider structure $S^{11}$ first. Each partner acts in full knowledge of both $x_1$ and $x_2$. His best action rule is obviously

$$\hat{\alpha}_i(x^i) = \hat{\alpha}_i(x_1, x_2) = \begin{cases} 1, & x_1 + x_2 \geqq 0 \\ 0, & x_1 + x_2 \leqq 0 \end{cases} \quad (i = 1, 2).$$

(This is analogous to the best action rule in $S^1$ in Example A.) Accordingly, partner 1 as well as partner 2 will have action $a_i = 1$ in the regions A, B, F of the $(x_1, x_2)$-plane, as shown on Chart 3.II (see upper panel and the panel marked $S^{11}$), and action $a_i = 0$ over all other regions. (Chart 3.II, page 22).

Hence

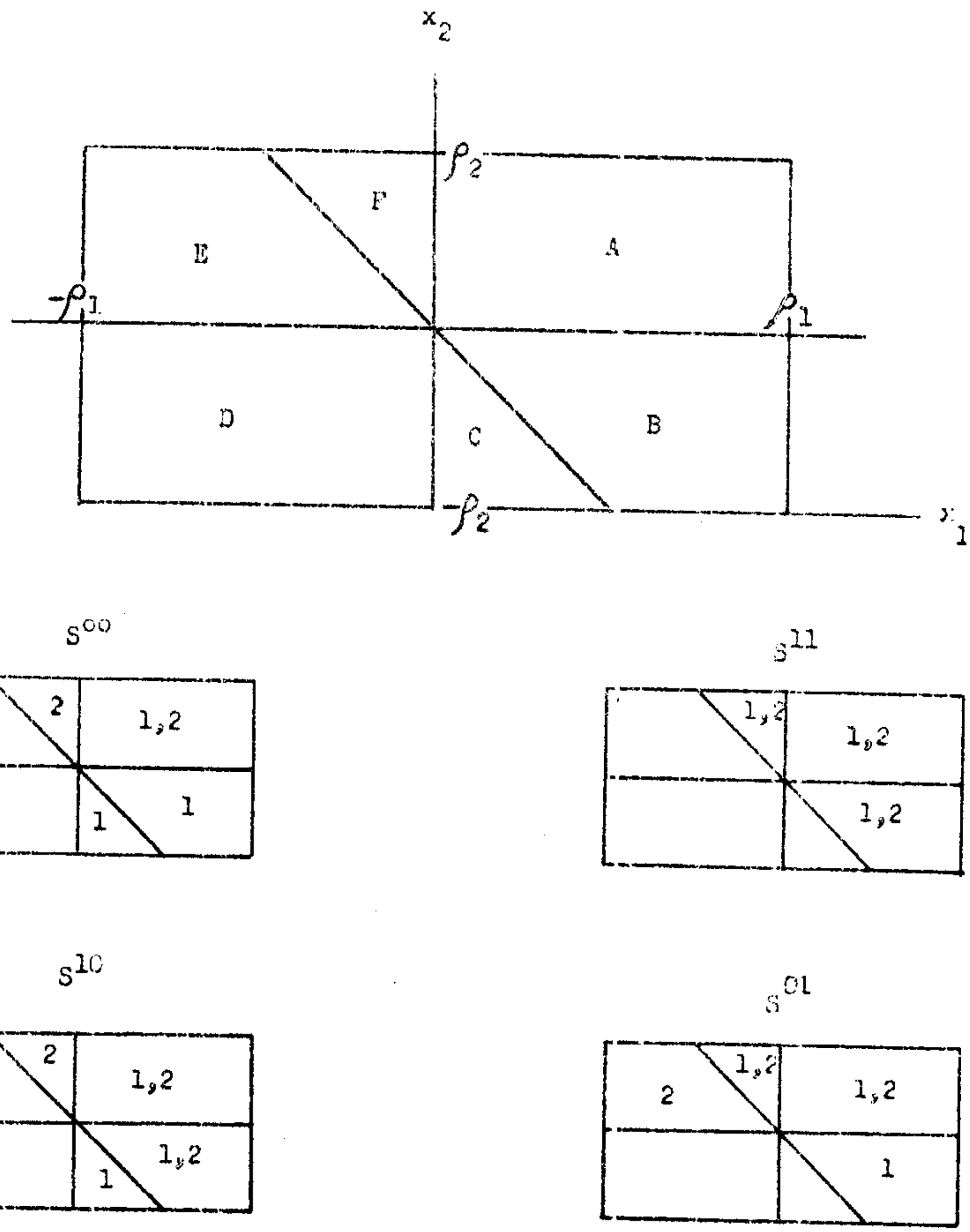
$$\hat{U}(S^{11}) = \int_{-\rho_2}^{\rho_2}\int_{-\rho_1}^{\rho_1} \frac{(x_1 + x_2)}{4\rho_1\rho_2}\, d\,x_1\, d\,x_2 = 2(U_A + U_B + U_F),$$

$U_A$, $U_B$, $U_F$ are the integrals over the regions A, B, F, respectively. In order to evaluate this expected payoff, and compare it with expected payoffs under rival structures, let us evaluate $U_A$, $U_B$, etc. (In all cases, the expression after the integral signs is $(x_1 + x_2)\, dx_1\, dx_2/4\rho_1\rho_2$, and will be omitted. Moreover, it will be convenient to introduce the "ratio between the two uncertainties,"

$$r = \rho_2/\rho_1 \leqq 1:$$

it will be seen that $\rho_1$ can be regarded as a scale factor.

Chart 3.II




The upper panel identifies 6 regions of the x-space.
The lower 4 panels identify the regions in which partner 1 or 2 takes non-zero action.

$$U_A = -U_D = \int_0^{\rho_2}\int_0^{\rho_1} = (\rho_1 + \rho_2)/8 = \rho_1(1+r)/8$$

$$U_B + U_C = -(U_E + U_F) = \int_{-\rho_2}^{0}\int_0^{\rho_1} = (\rho_1 - \rho_2)/8 = \rho_1\ (1-r)/8$$

$$U_F = -U_C = \int_0^{\rho_2}\int_{-x_2}^{0} = \rho_2^2/24\ \rho_1 = \rho_1 \cdot r^2/24; \text{ hence}$$

$$U_B = -U_E = \rho_1\left(\frac{1-r}{8} + \frac{r^2}{24}\right).$$

In $S^{oo}$, each partner i acts on the basis of $x_i$ only. Writing the payoff as

$$x_1\cdot(a_1 = 0 \text{ or } 1) + x_2\cdot(a_1 = 0 \text{ or } 1) + x_1\cdot(a_2 = 0 \text{ or } 1)$$
$$+ x_2\cdot(a_2 = 0 \text{ or } 1),$$

considering the actions of partner 1 first, we see that if $x_1 > 0$, the payoff is larger when $a_1 = 1$ than when $a_1 = 0$, for any given values (unknown to partner 1) of $x_2$ and $a_2$. Hence $\hat{\alpha}_1(x_1)=1$ for $x>0$. Similarly $\hat{\alpha}_1(x_1)=0$ for $x<0$. Finally, when $x_1=0$, the choice of $a_1$ (for given values of $x_2$ and $a_2$) does not affect the payoff. Similar results obtain for the second partner. Hence the best rule of action for $S^{oo}$ is

$$\hat{\alpha}_i(x_i) = \begin{cases} 1, & x_i \geq 0 \\ 0, & x_i \leq 0 \end{cases}, \quad i=1,\ 2.$$

This yields the markings given on Chart 3.II for the case $S^{11}$. Accordingly,

$$\hat{U}(S^{oo}) = 2U_A + U_B + U_C + U_E + U_F = 2U_A\ .$$

In $S^{10}$, partner 1 acts on basis of $x_1$ only but partner 2 has learned both $x_1$ and $x_2$ before acting. Therefore

$$\hat{\alpha}_1(x^{(1)}) = \hat{\alpha}_1(x_1) = \begin{matrix} 1, & x_1 \geq 0 \\ 0, & x_1 \leq 0 \end{matrix} \; ; \; \hat{\alpha}_2(x^{(2)}) = \hat{\alpha}_2(x_1, x_2) = \begin{matrix} 1, & x_1 + x_2 \geq 0 \\ 0, & x_1 + x_2 \leq 0 \end{matrix} .$$

$$\hat{U}(S^{10}) = 2U_A + 2U_B + U_C + U_F = 2U_A + 2U_B$$

For similar reasons,

$$\hat{U}(S^{01}) = 2U_A + U_B + U_E + 2U_F = 2U_A + 2U_F \quad .$$

We can now compute the (gross) advantage of a one-or two-way communication over no communication, neglecting the cost for the present:

$$\hat{U}(S^{10}) - \hat{U}(S^{00}) = 2U_B$$

$$\hat{U}(S^{01}) - \hat{U}(S^{00}) = 2U_F$$

$$\hat{U}(S^{11}) - \hat{U}(S^{00}) = 2U_B + 2U_F$$

Note that the (gross) advantage of the two-way communication over no communication is simply the sum of the (gross) advantages of the one-way communications in each direction. Moreover, if $r = \rho_2 / \rho_1 = 1$, it does not matter which of the two partners is the reporting one. But with $r < 1$, $S^{10}$ is preferred to $S^{01}$, i.e., it is advantageous to have reports sent by the partner whose observations vary over a wider range.

To compare the structures on the basis of net payoffs, $V(S)$, we have to subtract the corresponding fixed costs, which we can assume to be (with $\rho_1$ used again as a scale factor):

$$c(S^{00}) = 0 \; ;$$

$$c(S^{01}) = c(S^{10}) = k \cdot \frac{\rho_1}{12} \; , \; k > 0$$

$$c(S^{11}) = (k+h) \cdot \frac{\rho_1}{12} \; , \; h > 0 .$$

(If $h < k$ we have "decreasing marginal cost"; if $h > k$, "increasing marginal cost" of providing an additional direction of communication.) Denoting by $V^{ij}$ the maximum net payoff for structure $S^{ij}$ we have, apart from a positive proportionality factor ($\rho_1/12$),

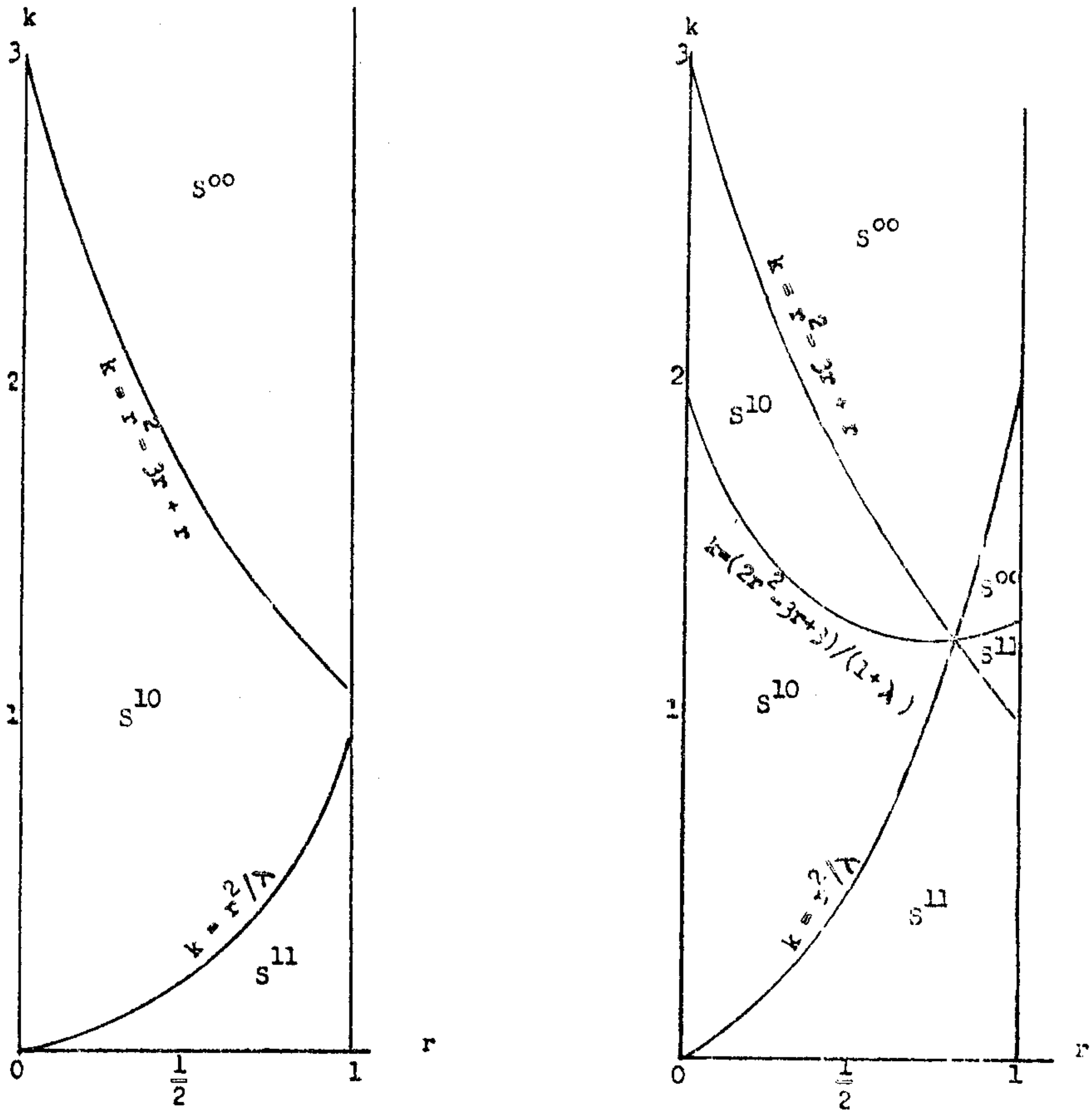
$$V^{10} - V^{oo} = r^2 - 3r + 3 - k$$
$$V^{11} - V^{oo} = 2r^2 - 3r + 3 - k - h$$
$$V^{11} - V^{10} = r^2 - h \quad .$$

Hence the preferred structure is

| | If $h > r^2$ | If $h < r^2$ |
|---|---|---|
| If $k < r^2-3r+3$ | $S^{10}$ | $S^{11}$ |
| If $k > r^2-3r+3$ | $S^{oo}$ | $S^{oo}$ if $h+k > 2r^2-3r+3$<br>$S^{11}$ if $h+k < 2r^2-3r+3$ |

Note that if $h > k$, then (since always $0 \leq r \leq 1$) the assumption $h < r^2$ contradicts the assumption $k > r^2-3r+3$. That is, the lower right-hand corner of the table just given becomes empty if the cost ($h+k$) of installing two-way communication is more than twice the cost ($k$) of installing one-way communication. This case ("increasing marginal cost of adding a communication direction") is represented on the left-hand panel of Chart 3.III, assuming a constant ratio $\lambda = h/k > 1$. The case ("decreasing cost") when $\lambda < 1$ is represented on the right-hand panel of Chart 3.III. With $\lambda$ fixed, the Chart divides the parameter space

Chart 3.III




Acceptance regions for three rival structures,

with $h/k = \lambda$, a constant.

$\lambda > 1$ ($\lambda = 9/8$)

$\lambda < 1$ ($\lambda = 1/2$)

$(r, k)$ in three acceptance regions, one for each of the three rival structures.

To sum up: the choice of best structure depends on three parameters: $r = \rho_2/\rho_1$, characterizing the range of $x_2$ relative to $x_1$; $k$, proportional to the cost of installing a one-way communication; and $\lambda = h/k$, the ratio of the cost of installing the second-way communication to the cost of installing one-way communication.

3.2. Alternative interpretation of cost.

As in Example A, we might consider the cost $c(S)$ not, or not only, as a fixed amount of money, but as the expense of the team's time that could otherwise be used for action. Let $(1-\gamma)$ be the proportion of time spent on reporting, $(1-\delta)$ the proportion of time spent on receiving a message, and $(1-\epsilon)$ the proportion of time spent on conferences (=two-way talks). Then the net payoff functions are

$$\text{In } S^{00}: \quad (a_1 + a_2)(x_1 + x_2)$$

$$\text{In } S^{10}: \quad (a_1 + a_2)(\gamma x_1 + \delta x_2)$$

$$\text{In } S^{01}: \quad (a_1 + a_2)(\delta x_1 + \gamma x_2)$$

$$\text{In } S^{11}: \quad (a_1 + a_2) \cdot \epsilon \cdot (x_1 + x_2)$$

$$V^{00} = \hat{U}(S^{00}) = 2U_A$$

$$V^{11} + (\gamma + \delta - 1)\,\hat{U}(S^{11}) = \epsilon \cdot (2U_A + 2U_B)$$

$$V^{10} = (\gamma + \delta)(U_A + U_B) + \gamma U_C + \delta U_F$$

For $S^{10}$, we have made use of Chart 2.II. We omit $S^{01}$ as it was already rejected in favor of $S^{10}$ upon comparing the gross payoffs. The problem consists in dividing the parameter space $(\gamma, \delta, \epsilon, r)$ into regions corresponding to the acceptance of $S^{00}$ or $S^{10}$ or $S^{11}$ as the best structure. We shall not pursue these computations here.

3.3. Operational problem for one-way communication structure.

To simplify the problem we shall assume $\rho_1 = \rho_2 = \rho$.

Suppose $x_1$ can be communicated to member 2. Let

$p(x_1) = \begin{matrix} 1 \\ 0 \end{matrix}$ according as $x_1$ $\begin{matrix} \text{is} \\ \text{is not} \end{matrix}$ reported.

$\alpha_1(x_1)$ be the rule of action for member 1

$\alpha_{12}(x_1, x_2)$ be the rule of action for member 2, in case $x_1$ is reported

$\alpha_2(x_2)$ be the rule for member 2 if $x_1$ is not reported.

The expected net payoff is (not subtracting structure cost, and denoting by $d$ the cost of sending a message):

$$(3.1) \qquad U = \frac{1}{4\rho^2} \int_{-\rho}^{\rho} \int_{-\rho}^{\rho} \Big\{ p(x_1) \left[ \alpha_{12}(x_1, x_2)(x_1 + x_2) - d \right] + (1-p(x_1)) \alpha_2(x_2)(x_1 + x_2) + \alpha_1(x_1)(x_1 + x_2) \Big\} dx_1 \, dx_2 \quad .$$

We first fix $p$, and find the best rules of action corresponding to $p$.

It is clear that

$$(3.2) \qquad \alpha_{12}(x_1, x_2) = \begin{cases} 1 \\ 0 \end{cases} \text{ as } (x_1 + x_2) \begin{matrix} \geq \\ < \end{matrix} 0$$

$$\text{or as } x_2 \begin{matrix} \geq \\ < \end{matrix} -x_1 \quad .$$

If in (3.1) we perform the integration with respect to $x_1$, the remaining integrand will be linear in $\alpha_2(x_2)$, the coefficient of $\alpha_2(x_2)$ being

$$\int_{-\rho}^{\rho} (1-p(x_1))(x_1 + x_2) \, dx_1$$

$$= x_2 \int_{-\rho}^{\rho} (1-p(x_1)) \, dx_1 + \int_{-\rho}^{\rho} (x_1(1-p(x_1)) \, dx_1 \quad ,$$

which is an increasing function of $x_2$. Hence for some number $\xi$, this coefficient will be positive for $x_2 > \xi$, and negative for $x_2 < \xi$. The best rule $\alpha_2$ is therefore given by:

(3.3) $$\alpha_2(x_2) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \text{ as } x_2 \begin{matrix} \geq \\ < \end{matrix} \xi$$

where $\xi$ is defined by:

$$\xi \int_{-\rho}^{\rho} (1-p(x_1)) \, dx_1 + \int_{-\rho}^{\rho} (x_1(1-p(x_1)) \, dx_1$$

(3.4) $$\xi = -\mathcal{E}(x_1 \mid p(x_1) = 0)$$

Turning now to $\alpha_1$, if in (3.1) we perform the integration with respect to $x_2$, the remaining integrand is linear in $\alpha_1(x_1)$, the coefficient of $\alpha_1(x_1)$ being

$$\int_{-\rho}^{\rho} (x_1 + x_2) \, dx_2 = x_1 (\rho + \rho) + \tfrac{1}{2} (\rho^2 - \rho^2) = 2x_1 \rho$$

Since this coefficient is positive or negative according as $x_1$ is positive or negative, the best rule $\alpha_1$ is given by

(3.5) $$\alpha_1(x_1) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \text{ as } x_1 \begin{matrix} \geq \\ < \end{matrix} 0$$

We now wish to find the best p. If in (3.1) we perform the integration with respect to $x_2$, the remaining integrand is linear in $p(x_1)$, the coefficient of $p(x_1)$ being:

$$p(x_1) = \int_{-\rho}^{\rho} [\, \alpha_{12}(x_1, x_2) \, (x_1 + x_2) - d - \alpha_2(x_2) \, (x_1 + x_2)] \, dx_2$$

Making use of (3.2) and (3.3) we can write (3.6):

$$(3.6) \qquad p(x_1) = \int_{-x_1}^{\rho} (x_1 + x_2)\, dx_2 \quad \int_{-\rho}^{\rho} d\cdot dx_2 - \int_{\xi}^{\rho} (x_1 + x_2)$$

$$= \int_{-x_1}^{\xi} (x_1 + x_2)\, dx_2 - \int_{-\rho}^{\rho} d\cdot dx_2$$

$$= x_1 ( \xi + x_1) + \frac{1}{2} ( \xi^2 - x_1^2) - 2d\cdot\rho$$

$$= \frac{1}{2} (2\xi x_1 + 2x_1^2 + \xi^2 - x_1^2) - 2d\cdot\rho$$

$$= \frac{1}{2} (x_1 + \xi)^2 - 2d\cdot$$

The roots of $p(x_1) = 0$ are

$$(3.7) \qquad \begin{cases} r_1 = -\xi - 2\sqrt{\rho d} \\ r_2 = -\xi + 2\sqrt{\rho d} \end{cases}$$

$p(x_1)$ is negative if $r_1 \leq x_1 \leq r_2$, and is positive otherwise. The best p is thus given by

$$(3.8) \qquad p(x_1) = \begin{cases} 0 \text{ if } r_1 \leq x \leq r_2 \\ 1 \text{ otherwise} \end{cases}$$

Equations (3.4), (3.7) and (3.8) jointly determine the optimal $\xi$ and p.

We can distinguish five cases regarding the position of $r_1$ and $r_2$:

| | | |
|---|---|---|
| I | $-\rho < r_1 < r_2 < \rho$ | : Two intervals of reporting |
| II | $r_1 \leq -\rho \leq r_2 < \rho$ | : One interval of reporting, on right |
| III | $-\rho < r_1 < \rho \leq r_2$ | : One interval of reporting, on left |
| IV | $r_1 = r_2$ | : Reporting on entire interval $[-\rho, \rho]$ |
| V | $r_1 \leq -\rho < \rho \leq r_2$ | : No reporting. |

Case I: By equation (3.4)

$$\xi = -\frac{r_1 + r_2}{2} ;$$

and, applying (3.7):

$$\xi = -\frac{1}{2}\left(-\xi - 2\sqrt{\rho d} - \xi + 2\sqrt{\rho d}\right) = \xi$$

Hence any value of $\xi$ such that Case I holds is optimal, with the corresponding $r_1$ and $r_2$ given by (3.7). Thus any interval of length $4\sqrt{c\rho}$ lying entirely within the interval $[-\rho, \rho]$ is optimal. Since the length of $[-\rho, \rho]$ is $2\rho$, this implies that Case I can hold only if $4\sqrt{\rho d} \leq 2\rho$ or

$$d < \rho/4 \quad .$$

Case II: By equation (4)

$$\xi = -\left(\frac{-\rho + r_2}{2}\right) = \frac{r_2 - \rho}{2}$$

Applying (7):

$$\xi = \frac{1}{2}\left(\rho + \xi - 2\sqrt{\rho d}\right)$$

$$\xi = \rho - 2\sqrt{\rho d}$$

and

$$r_2 = -\rho + 4\sqrt{\rho d} \quad .$$

These are the limiting values of $\xi$ and $r_2$ given by Case I if we let $r_1 = -\rho$.

Here it can be verified that $r_2 < \rho$ if and only if $d < \rho/4$.

Case III. By an argument similar to the above we see that Case III gives the limiting values of $\xi$ and $r_1$ given by Case I, if we let $r_2 = \rho$. Here $r_1 > -\rho$ only if $d < \rho/4$.

Case IV is obtained if we let d=0, and

Case V occurs if $d \geq \rho/4$.

In summary, we can describe the optimal operation as follows:

Optimal Operation

For all values of d,

$$\alpha_{12}(x_1, x_2) = {1 \atop 0} \text{ as } x_1 + x_2 {\geq \atop <} 0$$

$$\alpha_1(x_1) = {1 \atop 0} \text{ as } x_1 {\geq \atop <} 0$$

If $d < \frac{1}{4}\rho$, then any interval of length $4\sqrt{\rho d}$ lying entirely within the interval $[-\rho, \rho]$ is optimal as the set on which $p(x_1)=0$ (i.e., as the set on which no communication takes place. For any such interval $[r_1, r_2]$, the optimal $\alpha_2$ is given by:

$$\alpha_2(x_2) = {1 \atop 0} \text{ as } x_2 {\geq \atop <} -\frac{1}{2}(r_1 + r_2)$$

If $d \geq \frac{1}{4}\rho$, then

$$p(x_1) = 0 \text{ for all } x_1 \text{ in } [-\rho, \rho]$$

$$\alpha_2(x_2) = {1 \atop 0} \text{ as } x_2 {\geq \atop <} 0$$

3.4. Operational problem for two-way communication structure.

Again, only the case $\rho_1 = \rho_2 = \rho$ has been studied so far: we refer to the papers [3]-[8] on the "arbitrage problem;" in particular, Beckmann enunciated an approach similar to that used in Section 3.3 above for the one-way case, but did not obtain a solution. Kiefer and Orey [8] used a different approach and found a solution, having introduced the assumption that the operational rule is the same for both partners. A complete proof of the Kiefer-Orey solution for the general case is not available.[1]

1. Tompkins [9] studied the case when the $x_i$ are discrete, with uniform distribution. An exhaustive study for the case when $x_1$, $x_2$ each taking only two values, with any probability distribution, and with a somewhat generalised payoff function, has been made by Bratton in [10].

3.5. Complete, or constitutional, problem.

Clearly the solution of the complete problem for Example B will have to wait till the operational problem is solved not only for $S^{01}$ (Section 3.3) but also for $S^{11}$ — for example, till the Kiefer-Orey solution is proved to be generally valid.

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[9] C. B. Tompkins, "Notes on Computational Aspects of a Combinatorial Problem in Organization Theory, I, II, III. U. S. Department of Commerce, National Bureau of Standards, Los Angeles, September 1952-March 1953.

[10] D. Bratton, "The Arbitrage Problem where Each Partner Can Find Himself in One of Two Situations and Can Perform One of Two Actions," Cowles Commission Discussion Paper Economics 2075, May 13, 1953.

## 2.2. An alternative interpretation of structural cost.

We shall now modify our structural problem, by considering the fixed cost of the observation and communication facilities not as an amount (of money or utility) that is subtracted from the gross payoff, but as a portion of working time that is reserved for observations and communication and cannot be used for action. E.g., in our example of speculation, less time can be devoted to placing orders if more time is reserved for gathering and transmitting information. If $S^1$ is interpreted as a one-man team (Interpretation (I) above), the payoff function for both $S^1$ and $S^o$ becomes $u = \gamma \cdot a \cdot x$ where $1 - \gamma$ is the portion of time used for observations. If $S^1$ is interpreted as a two-men team -- one observer and one actor (Interpretation (II) above) -- the pure structural problem may consist in comparing such $S^1$ with a modified $S^o$, viz., with a team consisting of two men who act but neither observe nor communicate; such a comparison is equivalent to putting $1 - \gamma$ (the proportion of the team's time in $S^1$ used for observation and communication) equal $\frac{1}{2}$: a special case. Generally, $0 < \gamma < 1$.

We now have

$$\hat{V}(S^o) = |\mu|$$

$$\hat{V}(S^1) = \begin{cases} \gamma \cdot |\mu| & \text{if } |\mu| \geq \rho \\ \gamma \cdot \dfrac{\mu^2 + \rho^2}{2\rho} & \text{if } |\mu| \leq \rho . \end{cases}$$

The critical region favoring $S^1$ is now defined by

$$\gamma \cdot \frac{\mu^2 + \rho^2}{2\rho} > |\mu|, \quad |\mu| < \rho ; \quad 0 < \gamma < 1; \text{ that is,}$$

$$\frac{|\mu|}{\rho} + \frac{\rho}{|\mu|} > \frac{2}{\gamma}, \quad 0 < \frac{|\mu|}{\rho} < 1, \quad 0 < \gamma < 1. \tag{2.3'}$$

Therefore, given $\gamma$, the critical region depends on a single parameter, $|\mu|/\rho$.