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The Arbitrage Problem where Each Partner Can Find Himself in One of Two Situations and Can Perform One of Two Actions.

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1. Definition of the Problem.

J. Marschak has proposed the following problem (arbitrage problem where each partner can find himself in one of two situations and can perform one of two actions):

A firm consists of two partners who can confer only by telephoning. Each partner will find himself in one of two situations, the joint probability of the situations being known to the firm. In response to his situation, each partner can behave in one of two ways. Each of the sitteen combinations of situations and behaviors yields a known profit to the firm, to be described below. Rather than act on his own, either partner can call the other, at a cost γ to the firm, in which case, of course, the behavior can be decided jointly in view of the then known joint situation.

Given the cost γ of telephoning, the joint probability distribution on the situations, and the profit function, what is the best policy for the firm?

The profit function w is as follows:

	Second partner			
	1st situation		2d situation	
	1st action,	2d action;	1st action,	2d action.
First partner, 1st sit., 1st act.	$\rho_1+\rho_2+\sigma$	$\rho_1-\rho_2-\sigma$	σ	$-\sigma$
2d act;	$-\rho_1+\rho_2-\sigma$	$-\rho_1-\rho_2+\sigma$	$-\sigma$	σ
2d sit., 1st act.	σ	$-\sigma$	$-\rho_1-\rho_2+\sigma$	$-\rho_1+\rho_2-\sigma$
2d act.	$-\sigma$	σ	$\rho_1-\rho_2-\sigma$	$\rho_1+\rho_2+\sigma$

table 1

Here ρ_1, ρ_2, and σ are three numbers ≥ 0 such that $(\rho_1, \rho_2) \neq (0,0)$. It can be formulated

1) $\quad w = (\rho_1 a_1 + \rho_2 a_2)(x_1 + x_2) + \sigma a_1 a_2$

where $x_i = -\frac{1}{2}$ (resp. $+\frac{1}{2}$) when the i^{th} partner finds himself in the first (resp. second) situation; $a_i = -1$ (resp. $+1$) when the i^{th} partner performs the first (resp. second) action.

If the partners act without conferring, the firm gains $w(a_1, a_2, x_1, x_2)$; if one of the partners telephones, it gains $-\mathcal{T} + \max_{a_1, a_2} w(a_1, a_2, x_1, x_2)$

2. Intuitive Interpretation

Let us call the first and second situations faced by either partner "bad" and "good," $(x_i = -\frac{1}{2}, +\frac{1}{2})$, respectively, and characterize the two behaviors as "no action" and "action," $(a_i = -1, +1)$, respectively.

The second term ($\sigma\, a_1\, a_2$ of 1) represents a premium for concordance of actions, a penalty for discordance of actions.

Considering the first term $(\rho_1\, a_1 + \rho_2\, a_2)(x_1 + x_2)$ of 1), we see that if the situations of the partners are opposite, $x_1 + x_2 = 0$ and this term drops out. If on the other hand the situations are both bad, $x_1 + x_2 = -1$ and the first term of 1) is largest when neither partner acts--restraint is best in the bad situations. If both are good, the first term is largest when both act.

The relative contribution of each partner to the first term is governed by the ρ's, hence these two numbers express in a sense the relative strengths of the partners.

Let us suppose that the probability distribution is weighted toward similar situations for the two partners. Then each partner can deduce the other's situation from his own: there is no need to telephone. It is clear that each partner should then act in the good situation, not act in the bad situation.

At the other extreme, if the probability distribution is weighted toward opposite situations, then indifferently either a policy of never acting, or always acting, is clearly best, and each gains σ for the firm.

All the interest in the problem lies between these extremes, where there is uncertainty of the other partner's situation, and we wish to know when it is desirable to confer; and when not so, just what patterns of action are best.

Finally, note that if $\sigma = 0$, the problem becomes trivial. (cf. table 2, below; 12.12 is seen to be the only solution.)

3. Strategies.

A solution to the problem is of course a policy for the firm which specifies to each partner what he shall do in each of his possible situations. Since each partner has three actions ("don't act," "act," "confer"), there are $3^4 = 81$ possible policies. Each of these will be called a strategy, and noted by a sequence XY.ZW of four digits chosen out of the set 1, 2, 3, these representing the three actions in that order. The two pairs of digits refer to the partners, resp., and in each pair the first (resp. second) digit applies to the partner's first (resp. second) situation. Thus, e.g., 12.32 denotes the strategy "The first partner doesn't act if his situation is bad, acts if it is good; the second partner telephones if his situation is bad, acts if it is good." One notes that the result of this strategy is

$$2) \qquad (\rho_1 + \rho_2 + \sigma - \tau, \ -\sigma, \ \sigma - \tau, \ \rho_1 + \rho_2 + \sigma)$$

for the four joint situations bad-bad, bad-good, good-bad, good-good.

Let us denote by p_{ij} the probability that the 1st partner finds himself in his i^{th} situation, the 2d partner in his j^{th} situation, $(1 \leq i, j \leq 2)$.

In terms of the probability distribution $p = (p_{ij})$ and the structural parameters ρ_1, ρ_2, σ, and τ, one formulates the utility $U(S)$

of a given strategy S. To follow our example,

$$U(12.32) = p_{11}(\rho_1 + \rho_2 + \sigma - \tau) + p_{12}(-\sigma) + p_{21}(\sigma - \tau) + p_{22}(\rho_1 + \rho_2 + \sigma)$$

The 4-tuple 2) will be called the "utility coefficients."

To solve the problem is to find, for each value of the structural and probability parameters, those strategies which have the largest utility.

One strategy can be better than another a priori, i.e., independently of the probability distribution, or what amounts to the same thing, independently of which of the four joint situations prevails. Comparison of the utility coefficients thus enables us to weed out a large number of strategies. The results are as follows:

Table 2 - Efficient Strategies

	Strategy	Utility coefficients p_{11}	p_{12}	p_{21}	p_{22}
S_1	11.11	$\rho_1 + \rho_2 + \sigma$	σ	σ	$-\rho_1 - \rho_2 + \sigma$
S_2	12.12	$\rho_1 + \rho_2 + \sigma$	$-\sigma$	$-\sigma$	$\rho_1 + \rho_2 + \sigma$
S_3	13.11	$\rho_1 + \rho_2 + \sigma$	σ	$\sigma - \tau$	$\rho_1 + \rho_2 + \sigma - \tau$
S_4	11.13	$\rho_1 + \rho_2 + \sigma$	$\sigma - \tau$	σ	$\rho_1 + \rho_2 + \sigma - \tau$
S_5	22.22	$-\rho_1 - \rho_2 + \sigma$	σ	σ	$\rho_1 + \rho_2 + \sigma$
S_6	22.32	$\rho_1 + \rho_2 + \sigma - \tau$	σ	$\sigma - \tau$	$\rho_1 + \rho_2 + \sigma$
S_7	32.22	$\rho_1 + \rho_2 + \sigma - \tau$	$\sigma - \tau$	σ	$\rho_1 + \rho_2 + \sigma$
S_8	11.12	$\rho_1 + \rho_2 + \sigma$	$-\sigma$	σ	$-\rho_1 + \rho_2 - \sigma$
S_9	12.11	$\rho_1 + \rho_2 + \sigma$	σ	$-\sigma$	$\rho_1 - \rho_2 - \sigma$
S_{10}	12.21	$\rho_1 - \rho_2 - \sigma$	σ	σ	$\rho_1 - \rho_2 - \sigma$
S_{11}	21.12	$-\rho_1 + \rho_2 - \sigma$	σ	σ	$-\rho_1 + \rho_2 - \sigma$
S_{12}	12.22	$\rho_1 - \rho_2 - \sigma$	$-\sigma$	σ	$\rho_1 + \rho_2 + \sigma$
S_{13}	22.12	$-\rho_1 + \rho_2 - \sigma$	σ	$-\sigma$	$\rho_1 + \rho_2 + \sigma$

For each possible strategy, there exists one of the above ("efficient") strategies which is at least as good, independently of the values of the structural and probability parameters. (Of course, for certain values of these parameters, a strategy that is not listed may become equivalent to one of those listed--it would be too tedious to list such possibilities.)

Let us call a strategy concordant when in similar situations it produces similar behavior for the partners, i.e., when ρ_1 and ρ_2 appear, in each of its utility coefficients, with the same sign. The first seven of the strategies of table 2 are concordant. Each strategy involving communication (i.e., having a 3 in its code) is concordant, for by referring to table 1 we see that upon conferring, the partners do best to either both act or both not act, gaining $\rho_1 + \rho_2 + \sigma - \tau$.

4 - Nature of the solution

The relations $p_{ij} \geq 0$ $(1 \leq i, j \leq 2)$, $\sum_{ij} p_{ij} = 1$ on the coordinates of the probability distribution $p = (p_{ij})$ define a convex subset K of a four-dimensional vector space V. Suppose we fix the values of the structural parameters; the utility coefficients of each strategy being then determined, the utility of each strategy is a linear form defined in V. The condition that a given strategy S_i be best thus defines a convex polyhedron K_i contained in K. The solution to the problem is then defined by a partition of K into (thirteen in number, cf. section 3) convex (polyhedral) subsets: $K = \bigcup_i K_i$; in order that strategy S_i be best, it is nec. & suff. that $p \in K_i$.

Thus, for a fixed value of the structural parameters, one could compute the solution by solving thirteen systems of linear inequalities.

However, we choose to attack the problem first by a reduction of the parameters.

5 - Reduction of parameters; symmetries of probability space.

In the rest of this paper we shall restrict ourselves to the case

3) $$\sigma \geq \max(\rho_1, \rho_2)$$

The definition of the profit function (eq. 1) shows that an interchange of ρ_1 and ρ_2 simply interchanges the partners; it is thus no restriction to further assume

4) $$\sigma \geq \rho_1 - \rho_2 \qquad ,$$

a relation that implies

$$-\rho_1 + \rho_2 - \sigma \leq \rho_1 - \rho_2 - \sigma \leq \rho_1 - \rho_2 + \sigma \leq \rho_1 + \rho_2 + \sigma$$

Referring to table 2, one sees that these inequalities eliminate all the non-concordant strategies.

The function U, which associates to each strategy and value of the parameters the associated utility, defines the game up to equivalence. Since $\rho_1 + \rho_2 > 0$, the relation

5) $$U = \sigma + (\rho_1 + \rho_2)(p_{11} + p_{22} + U')$$

defines a new function U' varying monotonically with U, and hence defines a new game equivalent to the original. U' involves only two structural parameters.

Table 3. Utility Coefficients for U'

Strategy		p_{11}	p_{12}	p_{21}	p_{22}
S_1	11.11	0	0	0	-2
S_2	12.12	0	-a	-a	0
S_3	13.11	0	0	-2b	-2b
S_4	11.13	0	-2b	0	-2b
S_5	22.22	-2	0	0	0
S_6	22.32	-2b	0	-2b	0
S_7	32.22	-2b	-2b	0	0

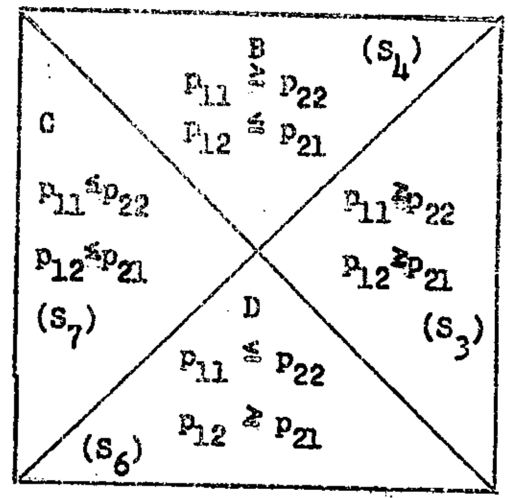


Fig. 1

where

6) $$a = \frac{2\sigma}{\rho_1 + \rho_2} \qquad b = \frac{\tau}{2\rho_1 + 2\rho_2}$$

We take account now of two symmetries $p_{11} \longleftrightarrow p_{22}$ and $p_{12} \longleftrightarrow p_{21}$ of the probability space. The inequalities $p_{11} \geq p_{22}$, $p_{12} \geq p_{21}$ and their opposites divide the space into four sectors, labelled A, B, C, and D as in figure 1. In respect to these sectors and symmetries we assert the following, where D_i denotes the region of probability space where S_i is a solution:

$D_3 \subset A$, $D_4 \subset B$, $D_7 \subset C$, $D_6 \subset D$; $D_1 \subset A \cup B$, $D_5 \subset C \cup D$. Any one of the four regions D_3, D_4, D_6, D_7, can be obtained from one of them by application of the symmetries; D_5 is obtained from D_1 with the symmetry $p_{11} \longleftrightarrow p_{22}$, and each is invariant under the symmetry $p_{12} \longleftrightarrow p_{21}$; D_2 is invariant under both symmetries.

These facts are verified as follows: $U_3' - U_4' = 2b(-p_{21} - p_{22} + p_{12} + p_{22}) = 2b(p_{12} - p_{21})$ (from table 3), hence $D_3 \subset A \cup D$, and $D_4 \subset B \cup C$. Also $U_3' - U_6' = 2b(-p_{21} - p_{22} + p_{11} + p_{21}) = 2b(p_{11} - p_{22})$, hence $D_3 \subset A \cup B$, $D_6 \subset C \cup D$. Combining these results, $D_3 \subset A$. Similarly for the others: D_4, D_6, D_7. D_1 and D_5 are located similarly from table 3; and the stated symmetries are immediate.

It results that a study of any one of the four sectors of fig. 1 is sufficient; the others being obtained from them by the symmetries. In sector A, e.g., only the three strategies S_1, S_2, and S_3 occur. We have thus eliminated all but three strategies.

6. Solution, in terms of extreme points.

Still with the assumption $\sigma \geq \max(\rho_1, \rho_2)$, and using the reductions of §5, we proceed to compute the solution, expressing the regions where each of the strategies is best by means of their extreme points, i.e., their vertices, in the probability space. (In that section we found that we could restrict ourselves to sector A: $p_{11} \geq p_{22}$, $p_{12} \geq p_{21}$, which simplifies the computation considerably.) In the course of computation three cases arise, depending upon the relative magnitudes of the structural parameters. The results are given in table 4; entered in this table are the points of the probability space which are the vertices of the convex region corresponding to each strategy.

Table 4 - Solution in terms of extreme pts.

Strategy	(I): $0 \leq b \leq \frac{a}{1+a}$.	(II): $\frac{a}{1+a} \leq b \leq 1$.	(III) $1 \leq b$.
S_1 11.11	(1000)(0100)(0010) v_1	(1000)(0100)(0010) v_5 v_6 v_7	Same as (II) with b = 1.
S_2 12.12	(1000)(0001) v_2 v_3 v_4	(1000)(0001) v_3 v_4 v_5 v_6 v_7	
S_3 13.11	(1000)(0100) v_1 v_2 v_3	(1000)(0100) v_3 v_6	
S_4 11.13	(1000)(0010) v_1 v_2 v_4	(1000)(0010) v_4 v_7	
S_5 22.22	(0100)(0010)(0001) v_1	(0100)(0010)(0001) v_5 v_6 v_7	
S_6 22.32	(0001)(0100) v_1 v_2 v_3	(0001)(0100) v_3 v_6	
S_7 32.22	(0001)(0010) v_1 v_2 v_4	(0001)(0010) v_4 v_7	

Here we have put

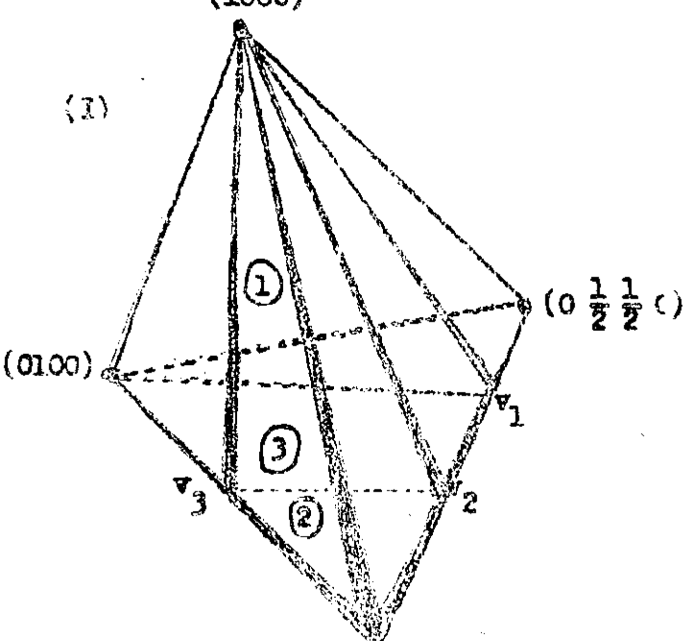
$$v_1 = \frac{1}{2}(b,\ 1-b,\ 1-b,\ b) \qquad v_5 = \frac{1}{2+2a}(a,\ 1,\ 1,\ a)$$

$$v_2 = \frac{1}{2a}(a-b,\ b,\ b,\ a-b) \qquad v_6 = \frac{1}{2ab+2b}(ab,\ 2b-a+ab,\ a-ab,\ ab)$$

$$v_3 = \frac{1}{2a+2b}(a,\ 2b,\ 0,\ a) \qquad v_7 = \frac{1}{2ab+2b}(ab,\ a-ab,\ 2b-a+ab,\ ab)$$

$$v_4 = \frac{1}{2a+2b}(a,\ 0,\ 2b,\ a)$$

Figures 2 & 3 show the result in sector A.



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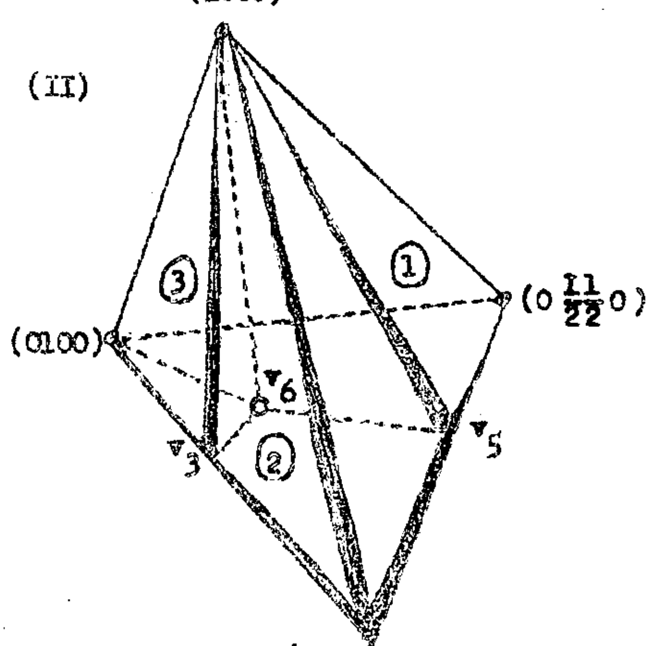


Fig. 3

7. Pictorial representation of the solution.

In this section we choose various values of all but two of the (both structural and probability) parameters, and plot the result in terms of the remaining two parameters. For this purpose we fix $\rho_1 = 1$, $\rho_2 = \rho \leq 1$, and choose new probability variables

$$x = p_{11} + p_{12} \qquad\qquad p_{11} = d + xy$$

$$y = p_{11} + p_{21} \qquad\qquad p_{12} = -d + x - xy$$

$$d = p_{11}p_{22} - p_{12}p_{21} \qquad\qquad p_{21} = -d + y - xy$$

$$p_{22} = 1 + d - x - y + xy.$$

The transformed range is defined by the relations

$$7)\quad \begin{cases} -\frac{1}{4} \leq d \leq \frac{1}{4}, \quad 0 \leq x,\ y \leq 1 \\ -\min(xy,\ 1-x-y+xy) \leq d \leq \min(x-xy,\ y-xy) \end{cases}$$

We will fix ρ, σ, γ, and d, at various values and plot the strategy-regions resulting in the $x - y$ square. The values chosen will fall in the case $\sigma \geq \max(\rho_1, \rho_2)$, hence the results of § 5 apply; in particular the decomposition into 4 triangles of the $x - y$ square by its two diagonals (fig. 4) is in accord with the decomposition of the original (p_{ij}) space (fig. 1)

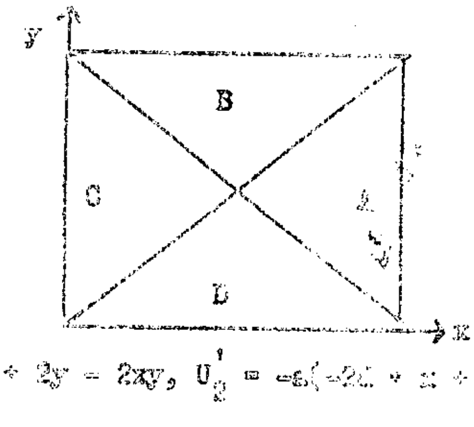
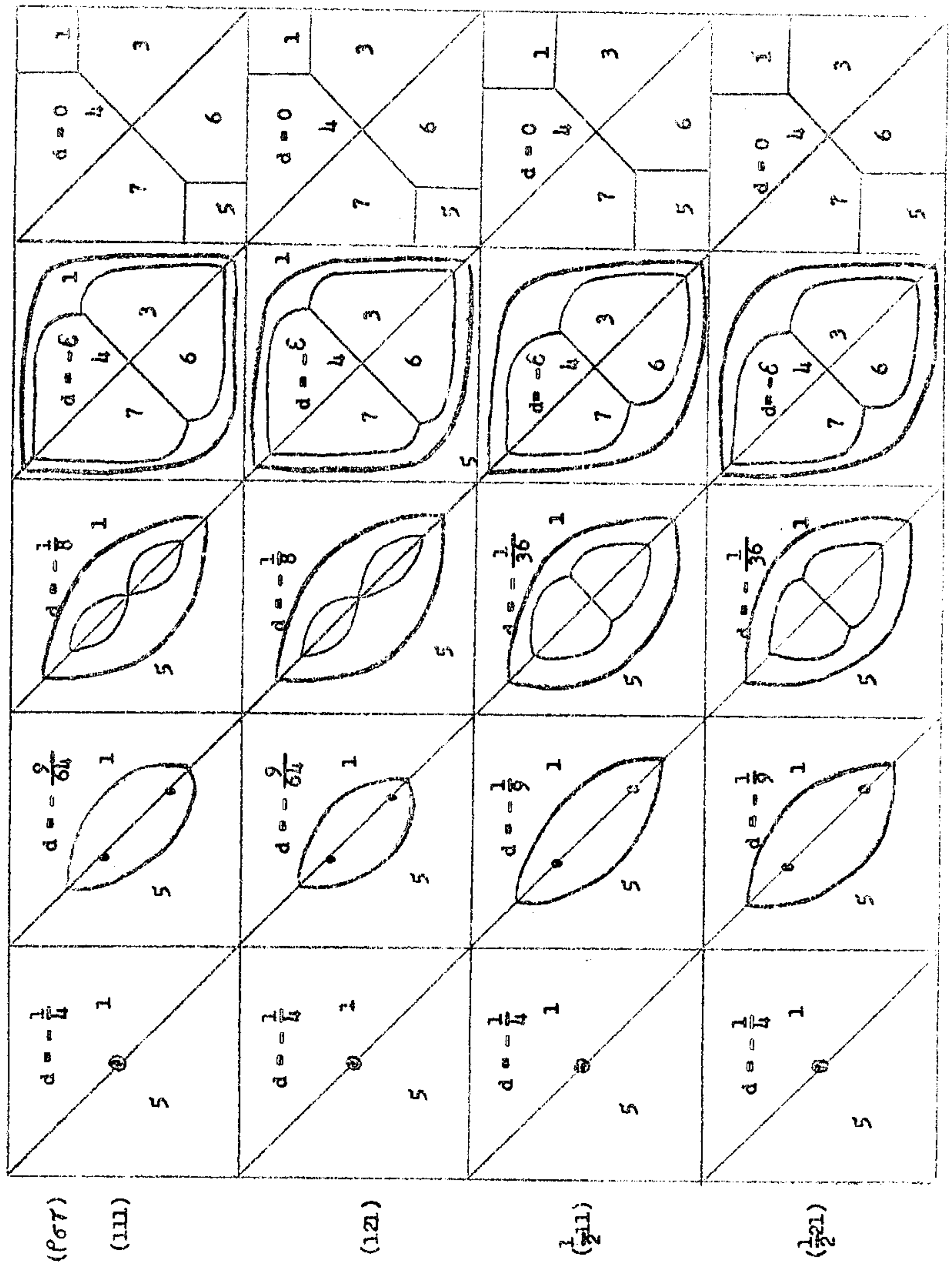
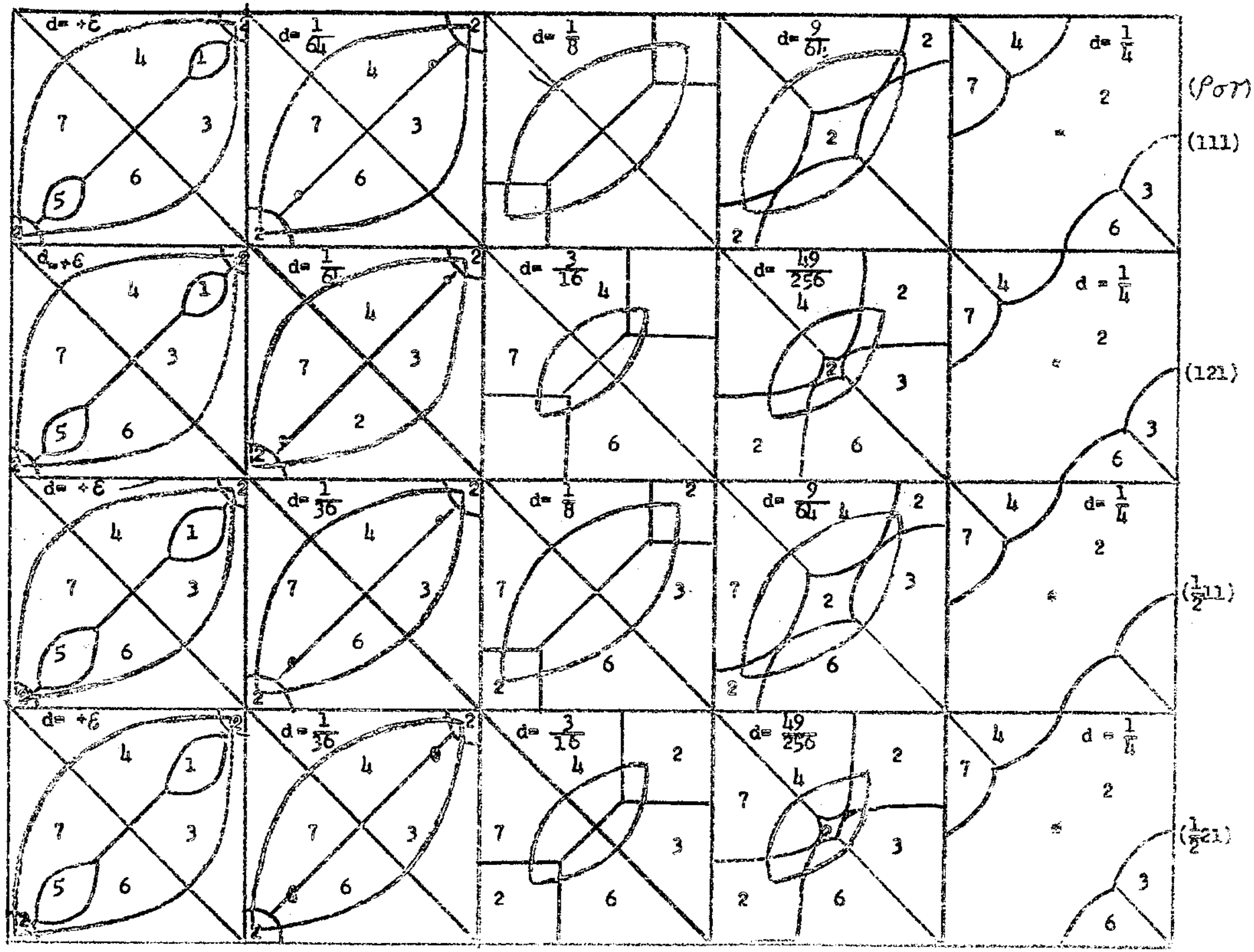


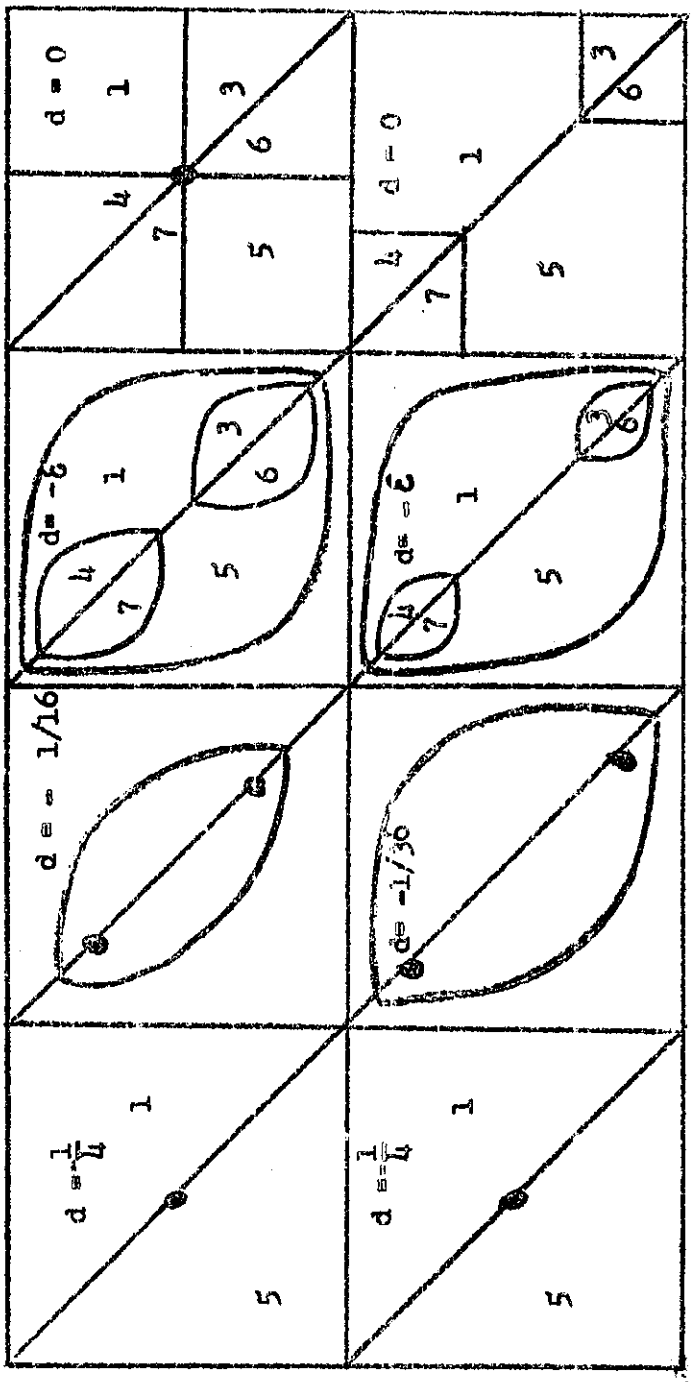
Fig. 4

Note that $U_1' = -2 - 2d + 2x + 2y - 2xy$, $U_2' = -a(-2d + x + y - 2xy)$, $U_3' = -2b(1 - x)$.

The pictures labelled $d = \pm \varepsilon$ show the tendencies as ε is small and positive. The strategy regions are labelled 1, 2, ... 7 corresponding to the strategies $S_1, S_2, \ldots S_7$. The red curves enclose the region defined by the relations 7). (For $d = 0$, this region is the whole square.)

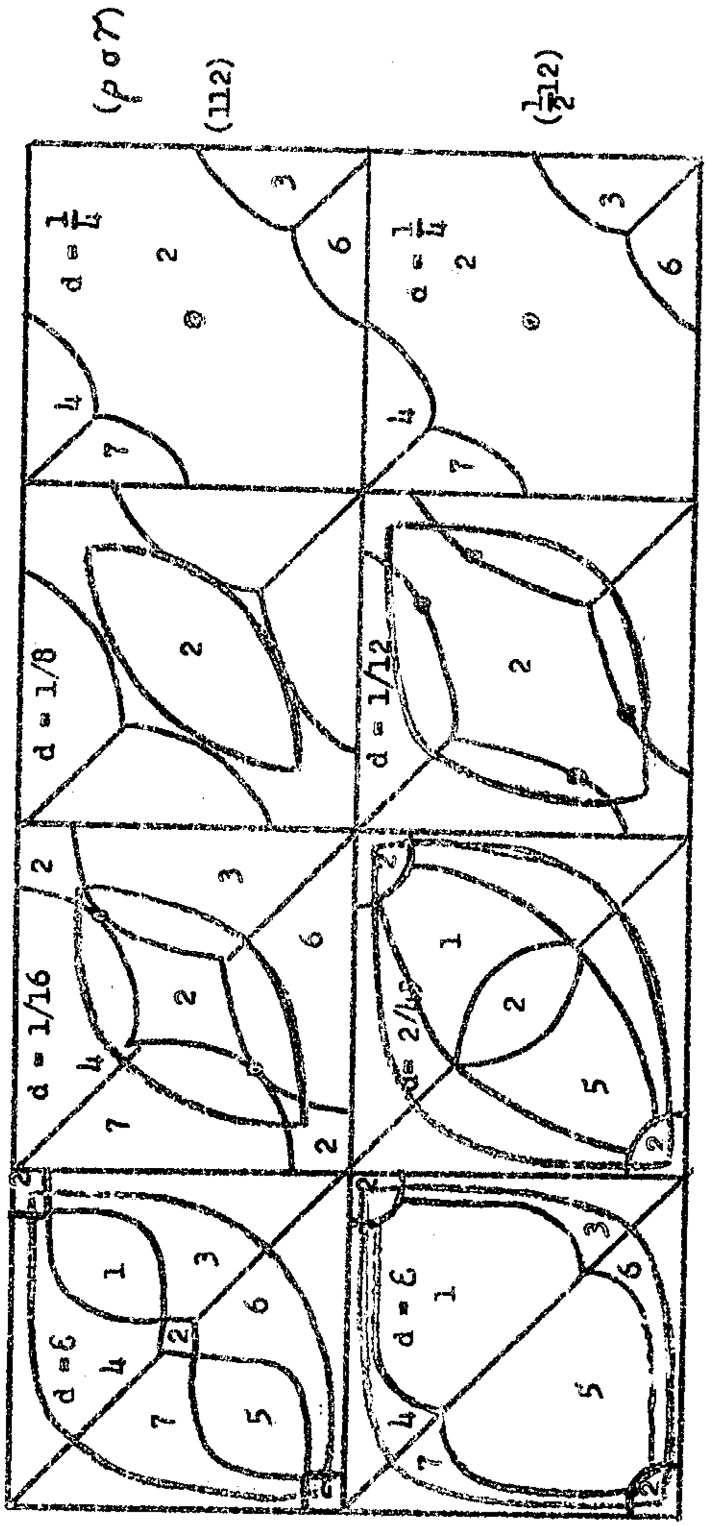

(ρστ)
(111)
(121)
(½11)
(½21)
d = -1/4
d = -9/64
d = -1/9
d = -1/8
d = -1/36
d = -ε
d = 0
1
3
4
5
6
7


(ρστ)
(111)
(121)
($\frac{1}{2}$11)
($\frac{1}{2}$21)
d= +ε
d= $\frac{1}{64}$
d= $\frac{1}{8}$
d= $\frac{9}{64}$
d= $\frac{1}{4}$
d= $\frac{3}{16}$
d= $\frac{49}{256}$
d= $\frac{1}{36}$



($\rho \sigma T$) (112)

($\frac{1}{2}$12)


(ρ σ 7)
(112)
($\frac{1}{2}$12)
d = $\frac{1}{4}$
d = 1/8
d = 1/16
d = ε
d = $\frac{1}{4}$
d = 1/12
d = 2/49
d = ε

8. General remarks on the solution

The following remarks of course apply only to the case $\sigma \geq \max(\rho_1, \rho_2)$, being derived from the results of §§ 5-7.

Let us group the seven strategies of table 3 into the following 3 classes:

(1) $S_1 = 11.11$, $S_5 = 22.22$. The "anti-correlation" strategies (because employed for negative values of d).

(2) $S_2 = 12.12$. The "correlation" strategy (because used when d is large and positive.)

(3) S_3, S_4, S_6, and S_7 (13.11, 11.13, 22.32, and 32.22). The conference strategies.

This grouping arises naturally from the symmetries discussed in § 5; when we restrict ourselves to one of the four sectors (fig. 1), in the competition between the strategies just one emerges from each of these groups. (In sector A these are S_1, S_2, and S_3.)

For negative values of d, S_2 drops out. $U_1 - U_3$ is independent of a, hence the solution is independent of a (i.e., σ):

<u>When the probability distribution exhibits a negative correlation between the partners, the solution is independent of σ.</u>

In fig. 2 § 6 we see that D_3 separates D_1 and D_2, hence in case (I) ($b \leq \frac{a}{1 + a}$) strategies S_1 and S_2 do not compete. In other words, if we suppress S_1 (resp. S_2), the competition of the remaining strategies determines the same region for S_2 (resp. S_1) as before. This can also be seen in the figures of § 7. Thus in case (I), D_1 depends only on b, D_2 only on $\frac{a}{b}$. From this one deduces that in case (c). S_1 drops out when $d > (\frac{b}{2})^2$; and hence the pictures in this range are determined by $\frac{a}{b}$ (i.e., by $\frac{\sigma}{\tau}$).

When $\tau \geq 2 + 2\rho$ $(b \geq 1)$, no communication takes place, as can be verified by §6.

§9 - Special case of equal probabilities.

Here we fix ourselves at the single point $p_{ij} = \frac{1}{4}$ $(1 \leq i, j \leq 2)$ of the probability space (case where each of the four joint situations is equally probable) and examine the solution in terms of the structural parameters. For this we take $\rho_1 = 1$, $0 \leq \rho_2 = \rho \leq 1$, $\sigma \geq 0$, $\tau \geq 0$. The solution can be expressed as a dissection of the $\sigma - \tau$ plane determined by a single point $(\sigma, \tau) = (\frac{1+\rho}{2}, 1+\rho)$. This result, together with the corresponding utilities is shown in fig. 5.

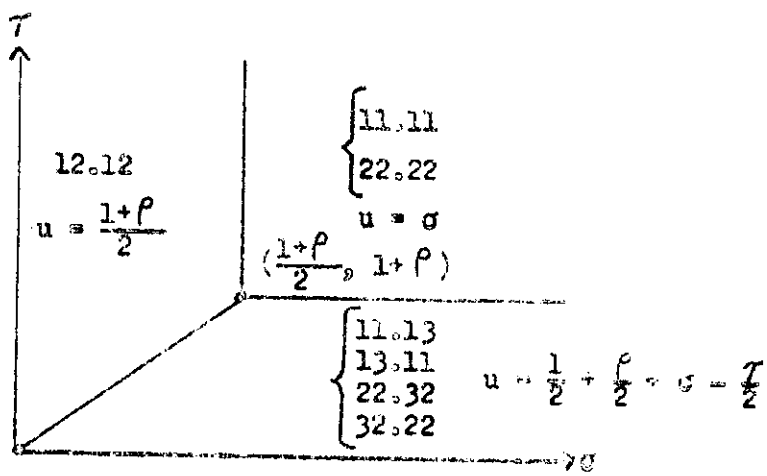


Fig. 5