COWLES COMMISSION DISCUSSION PAPER: ECONOMICS NO. 2073

The Mean Delay at A Traffic Intersection.

C. B. Winsten

May 6, 1953

The Mean Delay at A Traffic Intersection[1]

C. B. Winsten*

In a previous paper {Economics 2072} we formulated a model of the queue in the minor road at a traffic intersection. In that model it was found convenient to consider time as a discrete variable; with this convention the traffic stream in the minor road was supposed binomial. For the case for which the block-antiblock sequence was of general/geometric type, the mean waiting time per car was shown to be

$$\bar{z} = E\left\{\frac{q}{\alpha}\right\} = \frac{(1-\pi)\left\{E(b^2)+E(b)\right\}}{2\left\{1+(1-\pi)E(b)\right\}\left\{1-\alpha-\alpha(1-\pi)E(b)\right\}} \qquad 1.1.1$$

Here π is the parameter of the geometric distribution.

We have supposed the system to have settled down to statistical equilibrium; a necessary sufficient condition for this is that

$$1-\alpha \quad -\alpha(1-\pi)E(b) > 0. \qquad 1.1.2[2]$$

In this paper we calculate more detailed results. We assume the traffic in the major road is binomial, with parameter ρ.

We also assume that there is a constant safe interval, of length w time points, in front of each car (we used the symbol T in the previous paper.)

With these assumptions, it can be seen that the antiblock distribution is geometric, with parameter $\pi = 1-\rho$. Also we can find expressions for $E(b)$, $E(b^2)$ in terms of ρ and w, by methods similar to those used

---

[1] Research undertaken under contract between the Cowles Commission for Research in Economics and the RAND Corporation.

* On leave from the Oxford University Institute of Statistics

[2] This result follows immediately from Kendall [1] where a similar random walk is discussed.

in the continuous case [3,4]. Using these expressions in 1.1, we can obtain a formula giving mean waiting time as a function of α, ρ, and w. This function is plotted in the diagrams at the end of this paper.

2.1 The distribution of antiblock length.

We will suppose from now on that the time points are numbered in sequence $t = \ldots -1, 0, 1, \ldots$.

For shortness, if a car arrives at the intersection at the time point $t = t'$, we will say that there is a car at $t = t'$.

The safe interval w is supposed an integer and is defined as follows:

If there is a car at $t = t'$, then all the points t', $t' - 1$, $\ldots t' - w + 1$ are block points.

If in addition there is no car at any of the points $t' - w, \ldots, t' - 1$, then $t' - w$ is an antiblock point.

Suppose the point $t = 1$ is the first point of an antiblock; then there must be a car at $t = 0$, and no cars at $t = 1, 2 \ldots w$.

The point $t = 2$ is also an antiblock if and only if there is no car at $t = w$. Thus, given that $t = 1$ is an antiblock point, the probability that $t = 2$ is an antiblock point is $\pi = (1 - \rho)$.

Similarly, the probability that an antiblock will continue from $t = 2$ to $t = 3$ is π.

By induction, the probabilities of antiblock lengths of 1, 2, 3 ... are proportional to

$$1, \pi, \pi^2 \ldots$$

and so must be

$$(1 - \pi), \pi(1 - \pi), \pi^2(1 - \pi), \ldots$$

a geometric distribution.

3.1 The derivation of $E(b)$, $E(b^2)$

Now suppose $t = 1$ is the first point of a block. This means that there is a car at $t = w$, but none at $t = 0, 1, 2, \ldots w - 1$. The block will end at $t = w$ if there is no car at $t = w + 1, w + 2, \ldots, 2w - 1$. However, if the block does go on, suppose the next car after the one at $t = w$ is at $t = w + x_2$ say

$$1 \leq x_2 \leq w.$$

We can say that the second car dontributes a length x_2 to the length of the block.

We can also say that the first car always contributes a length $x_1 = w$ to the block.

If the block continues to a third car, we can consider the contribution x_3, of the third car to the block length, and so on.

3.2 We will now find the probability that the block is formed by just r cars.

If there are already $r - 1$ cars in the block then the probability that there will be no more is $\pi^w = P$. Thus the probability that the block will continue is $1 - P$.

Hence the probability that the block will contain just r cars is $(1 - P)^{r-1}P$.

3.3 If the block is formed by just r cars, the number of points it covers is $x_1 + \ldots + x_r$.

where $x_1 = w$

$1 \leq x_i \leq w$ $\qquad\qquad i = 2, \ldots r - 1.$

If the m.g.f of x_i is $\phi(u)$ say $(i = 2, 3 \ldots)$ then the m.g.f. of $x_1 + x_2 + \ldots + x_r$ is

$$\phi^{r-1} e^{w u}$$

and the m.g.f of block length if the block can contain any number of cars is

$$X(u) = \sum_{r=1}^{\infty} \phi^{r-1} e^{w u} P(1 - P)^{r-1}$$

$$= \frac{e^{w u} P}{1 - \phi(1 - P)} \qquad 3.3.1$$

It is also necessary to find $\phi(u)$.

$$\text{Prob}\left\{ x_i = x \;\middle|\; x_i > 0 \right\} = \frac{\pi^{x-1}}{\sum_{i=1}^{w} \pi^{x-1}}$$

so
$$\phi(u) = \frac{\sum_{x=1}^{w} e^{u x} \pi^{x-1}}{\sum_{x=1}^{w} \pi^{x-1}}$$

$$= \frac{1 - P e^{w u}}{1 - P} \quad \frac{1 - \pi}{1 - \pi e^{u}} \quad e^{u} \qquad 3.3.2$$

We have to find $E(b) = X'(0)$ and $E(b^2) = X''(0)$.

We obtain that

$$E(b) = X'(0) = \frac{1 - P}{(1 - \pi)P}$$

$$E(b^2) = X''(0) = \frac{1}{1 - \pi} - \left\{ \frac{3 - \pi}{(1 - \pi)^2} + \frac{2w}{(1 - \pi)} \right\} \frac{1}{P}$$

$$+ \frac{2}{(1 - \pi)^2 P^2}$$

so that the mean waiting time per car is, from 1.1.1.,

$$\bar{z} = \frac{1 - P(1 + w\rho)}{\rho(P - \alpha)} \qquad 3.3.3$$

where $P = \pi^{w} = (1 - \rho)^{w}$

and the necessary and sufficient condition for the system to settle down to statistical equilibrium is

$$\alpha < (1 - \rho)^{w}. \qquad 3.3.4.$$

The function given in 3.3.3 is plotted in the accompanying diagrams for various values of ρ, α and w.

4.1 Two way traffic in the main road.

The results apply if there is a two way stream in the main road, provided both streams are binomial. We assume the safe distance for crossing, w, is the same for both streams. If ρ_1, ρ_2 are the densities in the two streams, the probability that a time point will be occupied by a car in either of the streams is $\rho_1 + \rho_2 - \rho_1\rho_2$

and we can set ρ equal to this value in the formulas above.

5.1 We have not yet fitted these formulae to actual data, as we have not yet found any in a suitable form. It is therefore difficult to tell just how what the effects of the simplifications, we have made will be. However, from the data in [2] and [3] it can be seen that

(a) w is not constant, but varies fairly widely from car to car. However, the formula may still be fairly accurate if we use, the place of w, some parameter of the w distribution. Previous suggestions as to which parameter to use unclude the mean of the w distribution

(Greenshields [2]) and the median (Tanner [4]), but neither of these need necessarily be the best, as the following argument will show.

Consider the mean waiting time when α is small. In this case

$$\bar{z} = \frac{1}{\rho}(1 - \rho)^{-w} - \frac{1}{\rho} - w \text{ approx.}$$

$$= \frac{\rho}{2} w(w + 1) \text{ approx, if } \alpha \text{ is small (5.1.1.)}$$

When α is small, there is a negligable chance that a queue will form. Thus if w has a distribution, the mean waiting time can be obtained by averaging 5.1.1 over the distribution of w

$$\text{Then } \bar{z} = \frac{\rho}{2} E\left\{w(w + 1)\right\} .$$

suggesting that the constant $\bar{w}$ which will give the best fit is the solution of the equation $\bar{w}(\bar{w} + 1) = E\left\{w(w + 1)\right\}$.

(b) The assumption of a binomial distribution may break down in the major road. We would then have to formulate an alternative hypothesis. A reasonable one in some circumstances would be that the traffic distribution is the output of a queuing process. In this case the traffic would come in clusters, with a geometric distribution of time points between the clusters. Some of Greenshields data for congested roads suggests that this may be a useful approximation. $E(b)$ and $E(b^2)$ can be found for this type of traffic distribution by methods very similar to those used above. It has not been considered worthwhile to carry through these calculations for the present.

(c) The binomial assumption may break down in the minor road.

In this case, the analysis becomes very difficult. It is always possible to carry out experiments with various non-binomial sequences by Monte Carlo methods, however. We could, for example, transfer actual collected data onto punched cards, and find the waiting time distribution for the sequence, comparing it with the mean waiting time for the corresponding binomial sequence.

Bibliography

[1] Kendall, D. G.: "Some Problems in the Theory of Queues." *Journal of the Royal Statistical Society*, (Series B) 13, pp. 151-185 (1951).

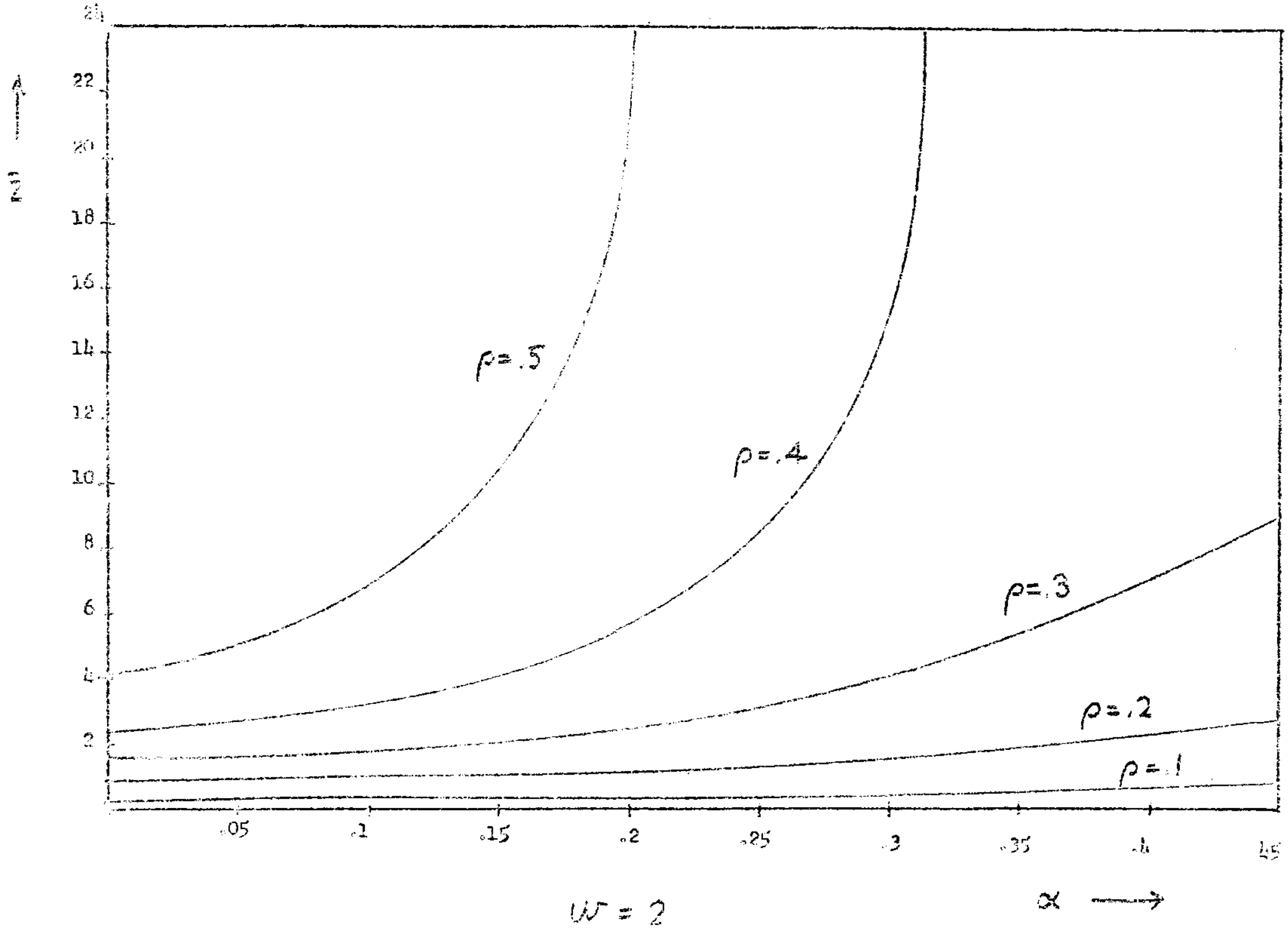
[2] Greenshields, B. D., Schapiro D. and Ericksen, E. L.: Traffic Performance at Urban Street Intersections," *Yale Bureau of Highway Traffic*, (1947).

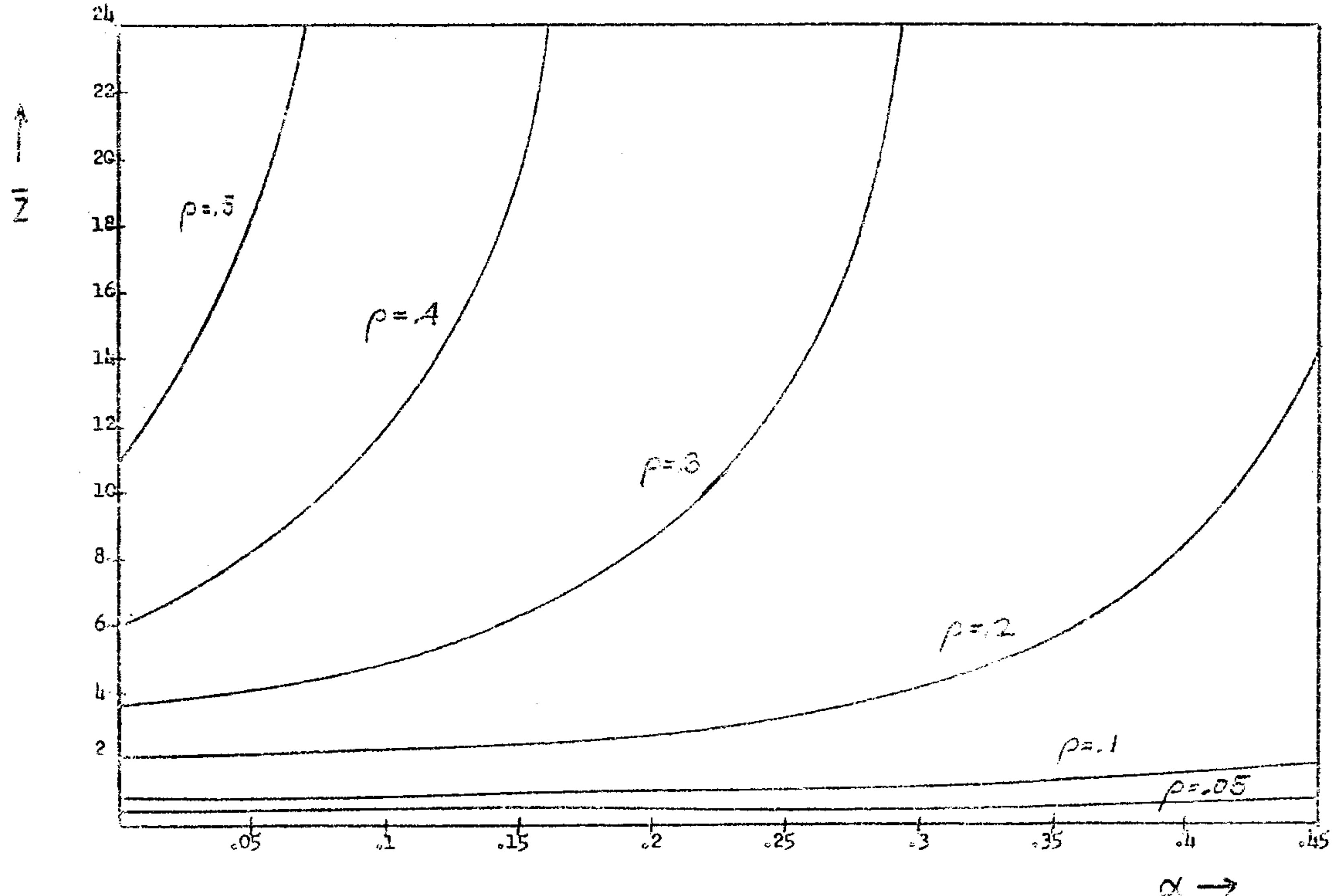
[3] Raff, M. S.: "A Volume Warrant for Urban Stop Signs," *Eno Foundation*, (1950).

[4] Tanner, J. C.: "The Delay to Pedestrians Crossing a Road." *Biometrika*, 38 pp. 383-392, (1952)

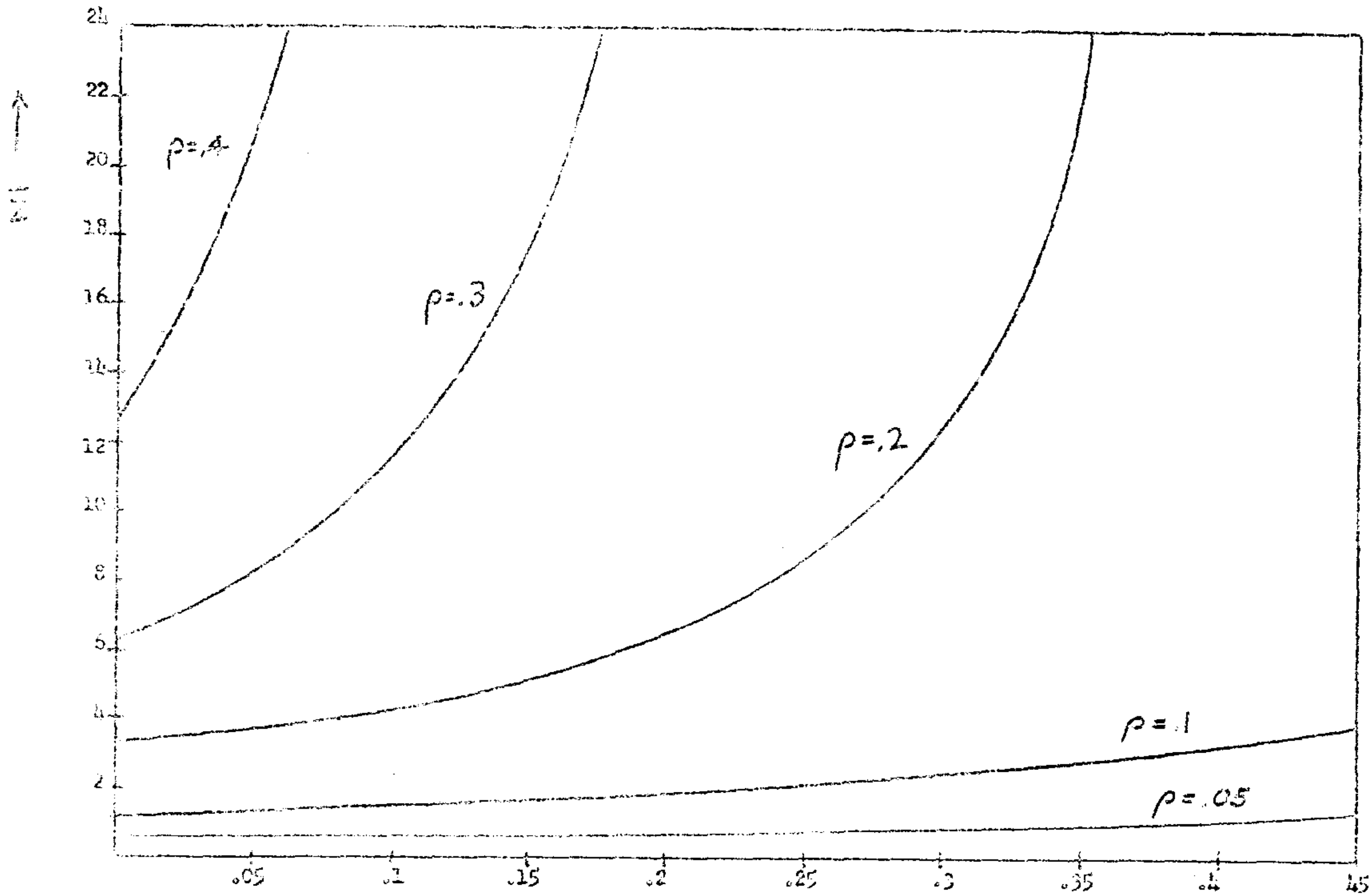
Acknowledgements:

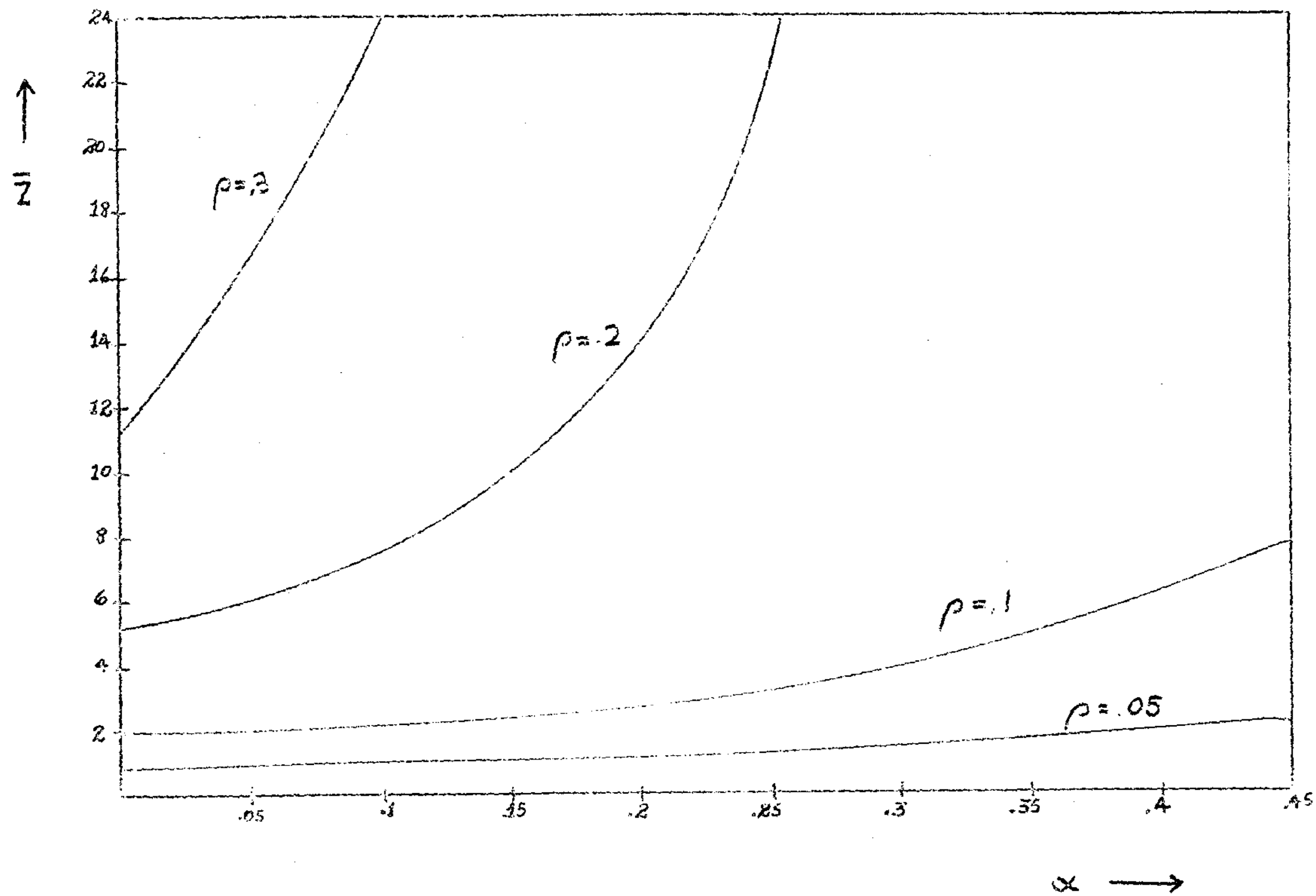
I must thank Mr. C. B. McGuire, who has cooperated throughout the present study, and the computing staff of the Cowles Commission for help in related calculations.


24
22
20
18
16
14
12
10
8
6
4
2
.05
.1
.15
.2
.25
.3
.35
.4
.45
ρ=.5
ρ=.4
ρ=.3
ρ=.2
ρ=.1
α
W = 2



$\bar{Z}$ ↑
24
22
20
18
16
14
12
10
8
6
4
2
ρ=.5
ρ=.4
ρ=.3
ρ=.2
ρ=.1
ρ=.05
.05
.1
.15
.2
.25
.3
.35
.4
.45
α →


ρ=.4
ρ=.3
ρ=.2
ρ=.1
ρ=.05
24
22
20
18
16
12
10
8
6
2
.05
.1
.15
.2
.25
.3
.35
.4
.45
α



$\bar{Z}$
24
22
20
18
16
14
12
10
8
6
4
2
$\rho = .3$
$\rho = .2$
$\rho = .1$
$\rho = .05$
.05
.1
.15
.2
.25
.3
.35
.4
.45
α

$\omega = 5$