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# Decentralized Resource Allocations

by

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## INTRODUCTION (MOTIVATION)

0. The main purpose of the Introduction is to explain the motivation for the selection of problems and the manner in which they are approached in the present study.

The reader should be warned that the formulation of the Introduction is heuristic rather than rigorous, though a more rigorous formulation is possible and planned for another occasion. The reader's indulgence is requested where the discussion covers well-known "classical" ground or goes in directions not obviously of relevance to the subject matter under discussion.

Also, a number of questions are left unanswered, some undoubtedly due to the writer's ignorance. Comments and criticism are, therefore, highly welcome.

1. The present study is intended as a contribution in the field of economic organization. Without attempting at this stage a rigorous definition of the latter concept, we may think of economic organization as a set of (legal or customary, say) rules imposed on human behavior.

The interest in alternative economic organizational structures[1/]

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[1/] Either or both adjectives preceding the term "structure" may be omitted when the context permits it.

is not new in economics, witness the classical debate between the doctrines of "laissez-faire" versus "interventionism," "capitalism" ("free enterprise") versus "socialism," or "centralization" versus "decentralization." 2/

In such debates, two types of questions arise 3/:

(a) the relative 4/ distribution of welfare of the different members of the community under alternative organizational structures;

(b) the extent to which the resource allocation induced by alternative economic structures tends to maximize the absolute level of welfare gives its relative distribution among community members.

Economics has traditionally (and especially so in recent times) focused its attention on the second problem ("(b)"), although there have been important attempts to develop the concept of a social welfare function which would encompass both ("(a)" and "(b)").

Those unwilling to postulate the existence of a social welfare function must content themselves with only a partial ordering of alternative positions of the economy 5/; consequently, they must

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2/ The terms in quotation marks have undergone considerable changes in their meaning; however, the subject of debate has always been the desirability of alternative economic organizational structures.

3/ Our formulation here uses terms such as "relative" and "absolute" welfare which smack of measurable and transferable utilities. In what follows rigorously defined counterparts of the terms used will be given in a manner not, involving either property.

4/ As between community members.

5/ The position of the economy is characterized by its production and consumption patterns and is, of course, to be distinguished from the economic structure which is an organizational concept.

be prepared for the possibility that their conclusions with regard to alternative economic structures will be only in the nature of a partial ordering.

2.1 In the present day state of thinking among economists, it seems natural to (partially) order the alternative positions of the economy in terms of the "Pareto-optimality" concept. According to the latter, a position $\underline{x}$ is (socially) "better" than $\underline{y}$ if no member of the community prefers $\underline{y}$ to $\underline{x}$ and at least one individual prefers $\underline{x}$ to $\underline{y}$. [6] A position $\underline{z}$ is then said to be Pareto-optimal (or, simply, optimal) if no other position is "better" than $\underline{z}$.

2.2.0 Clearly, given the technology and available resources, a position $\underline{x}$ may be Pareto-optimal for one set of individual preferences and not for another. On the other hand, a position $\underline{y}$ might be such that it could not be expected to be Pareto-optimal for any "plausible" (likely to occur) set of preferences. (Similarly with varying technologies, the preferences being fixed.)

If one believes to have more information about the "plausibility" rather than actual occurence of various preference patterns, it becomes worthwhile to distinguish positions according as to whether they are capable of being Pareto-optimal under a "plausible" set of preferences. Such positions may be called eligible (for Pareto-optimality) with regard to a given class C of preference and tech-

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6/ This formulation allows for the possibility that the individual orderings are partial.

nological condition. 7/

2.2.1 Koopmans' concept of efficiency may be regarded as an instance of "eligibility" for a particular class, say $C_o$, of preference and technological conditions, viz. when an increase in the availability of some goods without a decrease is any of the others is always regarded as an improvement. (See [2] pp. 59-60.)

3.1 The economic phenomenon "regulated" by the organizational structure is a process over time. Because of constant changes in the "environment" (technology, tastes, etc.) it is natural to assume that the economy may easily find itself at a non-optimal (or even non-eligible) position. It then becomes natural to appraise an organizational structure in terms of its tendency to push the economy toward a (steady state) optimal (or at least eligible) position. Ideally, one would then ask that the structure be such as to make (steady state) optimal positions into the (only!) equilibrium points of the dynamic process, that this equilibrium be stable and that the speed of convergence be maximized. 8/

When uncertainty is introduced into the picture, the concepts

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7/ This corresponds to Arrow's definition of "efficiency" in [1], p.

8/ These requirements are closely related to those of optimality interpreted as a criterion of ordering alternative time patterns as "points" in the commodity space with separate "coordinates" for each commodity at each point in time. The formulation of the text while somewhat less satisfactory on logical grounds has been chosen at this stage to simplify the exposition.

of equilibrium, stability, and speed of convergence must, of course, be properly defined, say in stochastic terms if probabilistic uncertainty is postulated. 9/

3.2.0 In observing the economic processes, one notes that certain activities would be unnecessary if everyone were omniscient. These information processing activities are quite likely to be among the fundamental difference of alternative economic structures 10/ and hence assume a basic role in our study.

It becomes worthwhile to study the "technology" of information processing. For consider two alternative organizational structures differing only with regard to the way in which information is processed; then it may well happen that one of them requires for information processing more of certain resources without requiring any less of others. If the "performance characteristics" (speed of convergence toward optimal equilibrium, etc.) are the same, it follows (under the usual assumption) that the structure requiring more resources is not "efficient" (in Koopmans' sense).

On the other hand, one must, of course, take into account the "performance characteristics" when the latter do differ.

3.2.1 The remarks of 3.2.0 concern a subject which, in principle, is well known to be of relevance in debates concerning problems of economic organization. As an example of such awareness we may mention

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9/ Again, this can be fitted into the concept of optimality in an appropriately defined space.

10/ Individual motivation aspects (e.g., the problem of "incentives") form another group of such differences.

the argument often raised in opposing administrative interventions in economic affairs (e.g., price control), namely that business firms are forced to spend money and energy on filling out numerous questionnaires and that government payrolls are increased through employment of individuals who must read such questionnaires. Lack of appreciation for the economic role of the merchant partly falls in the same category.

3.2.2 In fact, it is extremely difficult to appraise the resource requirements and "performance characteristics" of alternative information processing structures. For that reason, one is often satisfied with even fragments of information on this subject.

Even in fragmentary form this problem has been investigated almost exclusively against the background of a particular class of economic structures to which we may refer as that of <u>market structures</u>.

It has been my objective to develop a conceptual and analytical framework for consideration of a broader class of structures; an attempt in this direction forms a later part of this study. For this purpose it seemed essential that one should be able to formulate the theory of the market phenomena in a more abstract and generalized manner than had previously been done.

The need for generality in this context becomes particularly apparent when it is considered that dynamic and uncertainty aspects are essential for the problem at hand. 11/ The abstract approach,

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11/ I will concede that I had a prior independent interest in the possibility of extending the results of welfare economics and activity analysis to dynamic and probabilistic situations.

on the other hand, makes it easier to treat the phenomena of the market in what one might call "functional" (as distinct from "descriptive") manner which seems more suitable for later utilization in the comparative analysis of organizational structures.

Thus the emphasis in the early part of this study is on the market aspects of the economic process, with considerable generality as to the nature of the commodity space.

4.0 Let us now confine ourselves to problems arising in market structures.

The outstanding characteristic of these structures is that all individuals are assumed to act freely in what they regard as their best interest, except for actions interfering with the freedom of others. Thus, for instance, the use of violence is excluded and no one can be forced to trade with others if he does not wish to.

Let us define as <u>advantageous</u> an action (productive, exchange or gifts) which raises at least one person's utility [12/] and lowers nobody's.

We shall speak of a structure as <u>cooperative</u> [13/] provided equilibrium prevails if and only if no advantageous actions are possible for any group of individuals.

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12/ This term is used here to cover even those cases where there is no real-valued indicator and even if the underlying ordering is partial.

13/ A more fully descriptive term would be something like "self-interest cooperative" since it implies an effort toward utility maximization for every individual.

4.1 It is "almost self evident" [14] that in a cooperative market structure (I) an equilibrium position must be optimal; and furthermore, that (II) every optimal position, if attained, is an equilibrium position.

4.1.1 One might raise the question whether this formulation does not fail to take into account the possibility of the so-called external diseconomies, e.g., the factory smoke nuisance. To make our formulation correct we must do one of two things: either (a) treat the smoke as part of a deal which must be compensated for to be advantageous to smoke recipients, or (b) assume that all actions affect only those participating (i.e., absence of external economies or diseconomies).

4.2 The terms of exchange in the market may or may not be uniform.[15] Under the assumptions so far made there is reason to expect non-uniformity, if only since we have not excluded the type of preference function which would lead an individual A to want to give gifts to B but not to C.

We may find it useful at this stage to introduce the related concept of subjective virtual uniformity: it consists in the existence, for every pair of goods, of a ratio of exchange (relative price) at which every individual expects to be able to exchange arbitrary amounts of commodities held, a different ratio being im-

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[14] Cf. Wicksell, [ ], pp. 82-3.

[15] In the sense that, for any two goods, all exchanges are made in the same ratio.

possible for any amount. 16/ 17/

4.3 We may now note that assumption of cooperativeness (4.0) also had in it a subjective virtual element, since for equilibrium to prevail it was assumed necessary that no advantageous actions are (subjectively?) possible.

In particular, it may happen that when equilibrium is established it possesses the special property of subjective virtual uniformity, though in general this need not be the case; of course, if an equilibrium is established under conditions of subjective virtual uniformity, the cooperativeness assumption implies that it must be optimal, i.e., 4.1 (I) holds.

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16/ Hence two conditions are inherent: (1) every individual believes price to be independent of his (feasible) actions; (2) the prices expected by different individuals are the same. The second condition corresponds to absence of "discrimination." The first is usually considered to follow from the "atomistic" assumptions (cf. Pareto's behavior of type I, [ ], p. ).

17/ It may be noted that the usual proofs of non-optimality of resource allocation under monopoly (or monopsony) make essential use of the uniformity assumption.

Now where arbitrage is possible such uniformity will prevail, but it follows from 4.1. (I) that optimality is preserved since arbitrage does not isolate the postulate of cooperativeness. On the other hand where the uniformity assumption implies non-optimality, it must be that the requirement of uniformity contradicts the postulate of cooperativeness, i.e., there are still mutually advantageous actions possible (of a discriminatory type, of course). Hence it would seem that the "natural" (i.e., cooperative) course of events monopoly does not lead to non-optimality, unless its existence is due to "artificial" (e.g., legal) restrictions. (These conclusions do not follow if compensation is impossible.)

The condition of (subjective virtual) uniformity does, however, make a difference with regard to the validity of 4.1. (II). For it is conceivable that there might exist optimal positions which are not equilibrium points enjoying the property of subjective virtual uniformity. (We shall refer to the latter as uniform equilibria.)

As long as altruism, envy, etc., have not been ruled out from the preference functions, this possible failure of 4.1 (II) subject to the condition of uniformity is only natural (cf. 4.2 above).

However, such failures are also known to occur when the assumption of selfishness (or of "individualistic" preferences) is made, viz. that an individual's utility [18] does not depend on the possessions or consumption of others.

The classic example of such a failure is, of course, that of "increasing returns" where at the optimum uniform equilibrium is impossible.

On the other hand, it turns out that, subject to the assumption of selfishness, there exists an important class of cases where every optimal position can be regarded as one of uniform equilibrium.

[I.e., given that the economy is at an optimal point, there exists a set of uniform subjective virtual prices which would make selfish individuals want to stay where they are. It is important to note that in general, there will be other sets of price expectations producing the same result.]

This class is characterized by the convexity of certain sets

[18] Cf. footnote (12) in 4.0 above.

and functions, both on production and consumption side, e.g., non-increasing returns, non-increasing marginal substitution rates, etc.

4.4 Most recent studies, including this one, are concerned with the possibility of uniform equilibria for a market. 19/ Of course, when cases such as those of increasing returns are considered one must give up something to obtain a useful proposition different from 4.1 (II). Two approaches are possible. One is to require less than full uniformity, as above defined, but to retain some restriction on the terms of exchange. 20/

The other is to retain the uniformity requirement but to abandon the assumption 21/ that individuals will not leave any advantageous actions unexploited. 22/

4.5 The search for uniform cooperative equilibria at optimal points (or for something analogous when uniform cooperative equilibrium at an optimal is impossible) reveals a conviction that uniform cooperative equilibria have some highly desirable properties which are not necessarily present in (non-uniform) cooperative market equilibria whose existence at any optimum is guaranteed by 4.1 (II).

If we had a precise formulation of these properties, say $\left\{ P_1, P_2, \ldots, P_K \right\}$, we might try to see which ones could be present in any cooperative equilibrium when increasing returns prevail. Unfortunately, I have not been able to find in the existing literature any rigorous formulation

19/ Among the classics, Walras put emphasis on the uniformity condition in arguing the optimality of free competition, cf. , p.

20/ See Arrow and Hurwicz

21/ Implicit in what we have called cooperativeness!

22/ E.g., the principle that price should always equal marginal cost, even though monopoly prevails. (Cf. Marshall, , p. .)

of such a set of properties.

It seems fairly certain however, that these properties have a great deal to do with the efficiency of the information- processing aspects of the market phenomena.

In particular the desirability of uniformity appears to be related to the possibility of decentralizing the decision-making process. But even here, two important points require further study. First, the precise meaning of decentralization in this context. Second, the conceivable undesirability of decentralization in precisely those cases where uniformity fails (e.g., the case of increasing returns).

4.6 A further interesting characteristic of most contributions in the field is their tacit assumption of the existence of numéraire, since prices (and not merely exchange ratios) are used. From the viewpoint of the existence of uniform equilibria this assumption is unimportant and can be dropped. It is clear, however, that the existence of money (either as numéraire, or also medium of exchange) makes a great deal of difference in terms of the facility of communicating information. Yet here again rigorous treatment of the problem does not seem to have been undertaken.

The assumption of the existence of money as a medium of exchange is perhaps also implicit in that certain units (firms, resource owners) are maximizing their (money) profits, rather than utility directly.

5.0 So far we have been discussing the nature of the equilibrium point itself, but not the manner in which convergence toward equilibrium occurs. 23/ In this field even less is known. The results of Arrow and Hurwicz indicate those certain "natural" adjustment patterns may be conducive to stability. But I am not aware of the existence of a result showing that adjustment patterns which are (in some sense) "rational" from every individual's viewpoint produce stability of equilibrium.

5.1 Whether an individual is trying to move in the direction of an optimum, or trying to verify that his present position is optimal, there is an obvious advantage in having a criterion which is a <u>continuous</u> function of his position. For suppose that it is the individual's objective to maximize an entity (say, real-valued) whose variations over the (commodity or activity) space are quite arbitrary. Clearly, an inspection of all points of the space would be necessary. Even if only approximate maximization is required, nothing can be done without certain continuity properties.

This point is of importance in that it seems reasonable later to require that the price (regarded as a <u>functional operator</u> assigning to each point of the commodity space a certain real number, its "valuation") be not only <u>additive</u> (this corresponds to the uniformity requirement) but also <u>continuous</u>.

6.1 In setting up a model of the economy, as we shall subsequently, there arises the question whether the "activity" approach should be used. In it there exist activity variables, and points in the commodity space are selected indirectly through decisions concerning the activity variables.

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23/ Cf. Samuelson, ; Arrow and Hurwicz.

Since the preferences refer to the commodity space, it is possible to avoid any explicit treatment of the activities, confining oneself to the commodity space exclusively. The latter is the traditional approach and has recently been used by Arrow and Debreu, while Koopmans, Dantzig, Kuhn and Tucker have followed the former.

It should be noted that by appropriate formal devices it is always possible to go from the "commodity" to the "activity-commodity" language and vice versa. Hence what matters is primarily expediency and this tends to depend on the field of prospective application. In the present study both formulations are used.

6.2 The problem considered in section 4 above is usually treated on the assumption that all resources are owned by one of the individuals in the group for which Pareto-optimality is being achieved.

In general, this need not be the case: the resources do constitute a restriction on what is feasible, but there may not exist an individual owner whose utility is being maximized. It is here, however, that the results of (linear and non-linear) programming (whether "commodity" or "activity") theory are interesting, for they show that when resources are not owned individually, it may be expedient to postulate fictitious "custodians" and "managers," who behave like individual utility (profit) maximizers [24] (cf. Koopmans, [ ], pp. 93-95.

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[24] The recent contributions on the existence of competitive equilibrium points by Debreu and Arrow also go in this direction.

It may be noted that from the viewpoint of the general proposition of 4.1 the existence of unowned resources must be classified under "external diseconomies."

## Pricing in Additive Groups

2.1.3.2 As has been pointed out in the Introduction, Sec. 4.2, the case of uniform pricing is of particular interest to the economist, though the precise reasons for this interest do not seem to be well understood.

Now the function of the price, viewed as a functional operator, say $w^*$ on the space $W^+$ of commodity "bundles," is to associate with each commodity bundle [25] $w \in W^+$ a certain real number, say the "value" $w^*(w)$ of **w**.

A bundle **w** itself may be viewed as a collection of objects, in which case it is natural to define the operation of addition on $W^+$ by

$$w' + w'' = w' \cup w'' \qquad \text{for } w' \cap w'' = \Lambda$$

i.e., the sum of two disjoint bundles is their set-theoretic union. For the addition so defined, $W^+$ forms an additive commutative semi-group with the empty set as the identity element of addition.

On the other hand, we may dispense with the set-theoretic motivation and simply postulate that the commodity space is a semigroup of the above type.

Furthermore, instead of commodity bundles we may consider transfers of such bundles, say to or from a given person. Let $w \in W^+$ mean the transfer of **w** to that person, $-w$ the trans-

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[25] $w$ may be in $\mathcal{Y}$ or in $\mathcal{Z}$.

fer of $w$ from that person. Then the space $W$ of all transfers becomes an additive commutative group while $W^+$ plays the role of the non-negative "orthant."

Now the uniformity property of the price operator may be expressed as the additivity of $w^*$, in the sense that

$$w^*(w' + w'') = w^*(w') + w^*(w'')$$

for all $w', w'' \in W$.

Since the property of additivity of $w^*$ is only meaningful in virtue of the existence of addition in $W$ (or $W^+$), the problems of "uniform" pricing cannot well be discussed unless the commodity (commodity transfer) space is an additive commutative semi-group (group). 26/

Following customary usage we shall from now on refer to $W$ as the commodity space, omitting the word "transfer," while $W^+$ will sometimes be called the non-negative part of the commodity space.

Thus the assumption that $W$ is an additive group 27/ is in a sense minimal for the problem at hand, and it would be desirable to attack the problems of pricing for additive groups. So far this has not been done, the rigorous treatment being confined to various linear spaces which form a subclass of additive groups. The case of indivisible (discrete) goods indicates the need for going outside the class of linear spaces.

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26/ It can be shown that the (semi-) group properties, e.g., associativity and commutativity are inherent in the intuitive concept of uniformity as understood in economics.

27/ We shall understand that the addition is commutative.

Reasons can also be given why one should want $w^*$ to be a continuous function in a suitable topology.

Once W is assumed a topological space, there are grounds for requiring that W be a topological group, i.e., that addition be continuous. [28]

2.1.3.3 Convexity and separating hyperplanes in additive groups.

The question now arises how one should go about developing the theory for the general case of W assumed to be an additive topological group. Reflection indicates that the concepts of ordering, convexity, and hyperplanes, already available for linear spaces, must be generalized to groups of the type considered. Problems of ordering will be treated in a later section (3), while those of convexity and hyperplanes will now be discussed briefly.

We may start by stating the theorem which can be expected to be crucial in any extension of the results on existence of equilibrium (uniform) prices to the case of additive groups. This theorem, as stated by Klee ([ ], p. ) for linear spaces says roughly this: "Let (1) A and B be two convex sets in a linear space $\mathcal{X}$ with interiors, in any, disjoint; suppose further, that (2) either (a) $\mathcal{X}$ is finite-dimensional or (b) at least one of the sets A, B possesses an interior. Then there exists a hyperplane separating A from B." [29]

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28/ The following facts are worth nothing. Let W be a linear normed space and let $w^*$ be additive. Then the continuity of $w^*$ implies its (real-) homogeneity and boundedness; also, if $w^*$ is isotone and $W^+$ possesses interior, it follows that $w^*$ is continuous.

29/ Arrow's treatment in [ ], can be extended without difficulty to topological linear spaces since the above theorem is virtually all that is necessary. This is done elsewhere in this study.

We have underlined the terms which, for our purposes, must be "reasonably" defined for groups in such a manner as to coincide with the usual definitions when applied to linear spaces. We shall suggest such definitions in what follows.[30]

I. <u>Convexity</u>. (a) If G is an additive group and $x \in G$, we write 2x to mean x + x and, in general, mx to mean x + x + ... + x (<u>m</u> terms).

(b) Let $x', x''$ be elements of G. Define the set $S_{x',x''}$ of all elements $x'''$ such that

$$x''' \in G$$

and

$$\alpha' x' + \alpha'' x'' = (\alpha' + \alpha'') x'''$$

for some real numbers $\alpha'$, $\alpha''$ with

$$\alpha' \geq 0, \alpha'' \geq 0, \alpha' + \alpha'' > 0 .$$

Let $\bar{S}_{x',x''}$ denote the closure of $S_{x',x''}$. ($\bar{S}_{x',x''}$ is the analogue of the <u>segment</u> connecting $x', x''$.)

(c) A <u>set</u> $A \subseteq G$ is said to be <u>convex</u> if and only if with any two points $x', x''$ it also contains the "segment" $\bar{S}_{x',x''}$.

(This definition does not imply that a convex set must be closed).

II. <u>Hyperplane</u>. As in linear spaces, we may define a hyperplane as a set

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[30] I have profited from discussing the convexity concept with Professor Reinich of the University of Michigan and Professor Rosenbloom at the University of Minnesota.

$$\left\{ w \in W \ : \ w^{*}(w) = \gamma \right\}$$

where $w^*$ is an additive continuous functional and $\gamma$ a fixed real number.

2.1.3.4 Suppose now that with definitions such as those in 2.1.3.3 or analogous, the separating hyperplane theorem is valid. 31/ One may conjecture that the usual theorems on the existence of uniform prices at efficient or Pareto-optimal points would follow. 32/ Thus it seems extremely desirable to encourage investigations as to the validity of the separating hyperplane theorems in additive topological groups.

2.1.3.5 It should be clear, however, that such a theorem would not establish the possibility of a uniform price system where indivisibilities are not only present but are also assumed to imply lack of the needed convexity properties (e.g., increasing returns).

In fact, it would be of interest to see to what extent the reasons given for the plausibility of the convexity assumptions for (finite-dimensional) linear spaces 33/ retain their validity when applied to more general additive groups.

2.1.3.6 Going in the "other direction," we may be interested not only

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31/ It might be necessary to introduce some additional hypotheses for groups which are not linear spaces.

32/ We may note that the converse theorem (viz. that a point of competitive- or saddle-point-equilibrium- is efficient or Pareto-optimal) can immediately be seen to be valid in additive groups. The proof, of course, does not use the separating hyperplane theorem. Cf. 4.1 (I) in the Introduction, as well as the remarks concerning 4.1 (I) in sec. 4.3 of the Introduction.

33/ Cf. Koopmans as quoted by Arrow, [ ], p.

in various convexity properties, but may want to have some <u>differentiability</u> properties at our disposal.[34] These are obtainable if we assume the relevant linear topological spaces to be Banach spaces and use the concepts of the Fréchet (or Gâteaux) differentials.[35](+)

Elsewhere in this study we present some results of this type. In the writer's opinion, they are of interest from the viewpoint of application when one wishes to go beyond mere existence theorems. This would be especially the case if dynamic problems were to be considered.

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34/ These are of particular value when the usual convexity postulates fail.

35/ I do not know whether a more general theory of this type exists.

(+) Cf. Hille, [ ], p.