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# On the Arbitrage Problem[1]

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1. Introduction. The purpose of this memorandum is to give a solution to a problem posed by Marschak in CCDP 2029; the problem will be solved under a restriction stated three paragraphs below, the solution being stated in the last paragraph of the paper. The lacunae in previous attempts at solving the problem will also be pointed out.

Marschak's problem was to find rules of action for two members of a firm which will maximize the firm's profit, under the following conditions: $x_1$ and $x_2$ are independent chance variables, each with rectangular distribution on the unit interval. Member number 1 observes $x_1$ and member number 2 observes $x_2$. Each member then decides to commit the firm, not to commit the firm, or to phone the other member and then decide whether or not both should commit, the cost of phoning being $c > 0$ (2c in some of the CCDP's, c in others) if either or both decide to phone. The revenue to the firm is $x_1 + x_2 - 1$ for each decision to commit and 0 for each decision not to commit (thus, it is obvious what to do if one has already phoned). The problem is to give rules of action

1. Reproduction by the Cowles Commission of a paper written under a research contract with the Office of Naval Research, F. D. Rigby, Scientific Officer. The solution given here was communicated by the authors to M. Beckmann and J. Marschak in August 1952 and was used in Beckmann's Cowles Commission Discussion Paper Economics 2058. The present paper gives the proof.

for the two members which maximize expected profit (= expected revenue minus expected cost) for the firm.

Marschak conjectured in CCDP 2034 that any optimal rule would be characterized by numbers $0 \leq \alpha_i \leq 1 - \alpha'_i \leq 1$ $(i = 1,2)$ such that member i does not commit, phones, or commits without phoning, according to whether $0 \leq x_i \leq \alpha_i$, $\alpha_i < x_i < 1 - \alpha'_i$, or $1 - \alpha'_i \leq x_i \leq 1$, respectively. It was also conjectured in CCDP 2034 that $a_1 = a'_1 = a_2 = a'_2$. We shall see that this conjecture is not generally true. Beckmann showed in CCDP 2034 that any optimum procedure can be described, for each member, by a (possibly degenerate) sequence of five intervals, where one phones, does nothing, phones, commits, or phones as $x_i$ increases from 0 to 1. Beckmann also proposes a particular family of solutions to the problem, which our results show not to be a solution except in the trivial case when c is very large. The mistake here is that the conditions of Table 1 or 2 of that paper are necessary but not sufficient for a solution to be optimum: these conditions insure that either member's strategy is best relative to the fixed value of the other's; this does not entail that the pair is best among all pairs. Faxén also tried to compute the optimum scheme in CCDP 2037. His result differs from that which we obtain below, the reason being that his Case III which he does not consider is precisely our optimum one for small c. It is easy to see that the "solutions" of Beckmann and Faxén cannot be optimum for sufficiently small c: the plan for which (in the terminology of the next paragraph) $C_i$ is empty and $N_i = \left\{x_i \mid x_i \leq \frac{1}{2}\right\}$ has $I = 0$ and $A = \frac{1}{4}$, while that cited has $I > 0$ and $A \to 0$ as $c \to 0$.

<u>We now solve the problem under the following restriction</u>: the length of the interval in which member 1 commits equals the length of the interval in which member 2 commits, and the length of the interval in which 1 does nothing

equals that in which 2 does nothing. We find the optimal policy satisfying the above restriction. It seems plausible that this is indeed the optimal policy.

The problem can be described as follows: let $C_i$ and $N_i$ denote, respectively, the sets of $x_i$ where member i commits or does nothing. Let P and Q denote, respectively, the subsets of the unit square where $x_1 + x_2 - 1$ is positive or nonpositive. Let $A(C_1, N_1, C_2, N_2) = (b_1 + m_1)(b_2 + m_2)$, where $b_i$ and $m_i$ are the lengths of $C_i$ and $N_i$, respectively. Let

$$J(C_i,N_j) = \int_{N_j x C_i} |x_1 + x_2 - 1| \, dx_i \, dx_j \qquad \text{for} \quad i \neq j \;;$$

$$J(C_1,C_2) = 2 \int_{(C_1 x C_2) \cap Q} |x_1 + x_2 - 1| \, dx_1 \, dx_2 \;;$$

$$J(N_1,N_2) = 2 \int_{(N_1 x N_2) \cap P} |x_1 + x_2 - 1| \, dx_1 \, dx_2 \;.$$

We use $C_1 \times N_2$, $C_1 \times C_2$, $N_1 \times C_2$, $N_1 \times N_2$, to refer to the rectangles in the unit square with the appropriate sides. Let $I(C_1, N_1, C_2, N_2) = J(C_1, N_2) + J(C_2, N_1) + J(C_1, C_2) + J(N_1, N_2)$. The gain in the expected income to the firm from using a given $C_1$, $N_1$, $C_2$, $N_2$ over that from having both members always phone is

$$cA(C_1, N_1, C_2, N_2) - I(C_1, N_1, C_2, N_2) \;,$$

and our problem is to maximize this quantity. Our solution will proceed by finding for each fixed value of A the $C_i$ and $N_i$ which minimize I(Section 2) and by then using this to find for any c the $C_i$ and $N_i$ which maximize cA - I (Section 3).

2. Minimum I for fixed A. We suppose that $b_1 = b_2 = b$, $m_1 = m_2 = m$. Of course, b and m are $\geqslant 0$ and $b + m \leqslant 1$. We also assume $b_i \leqslant m_i$ without loss of generality: any solution obtained under this restriction corresponds in an obvious manner to another solution obtained by replacing $x_i$ by $1 - x_i$ (i=1,2). Finally, we may assume $b + m \geqslant \frac{1}{2}$: any procedure violating this restriction is inferior to that for which $N_i = \left\{x_i \mid 0 \leqslant x_i \leqslant \frac{1}{2}\right\}$ and $C_i$ is empty, the latter procedure giving $A = \frac{1}{4}$, I = 0.

We now note several simple facts. $\int_R |x + y - 1|\,dx\,dy$ is the first moment of R around $x + y - 1 = 0$ multiplied by $\sqrt{2}$. The first moment multiplied by $\sqrt{2}$ of an isosceles triangle of side a about its hypotenuse, is $\frac{1}{6}\ a^3 = g(a)$ (say). Hence, if $m \geqslant \frac{1}{2}$ and $N_i = \left\{x_i \mid 0 \leqslant x_i \leqslant m\right\}$ for i = 1,2, then $J(N_1,N_2) = 2g(m - \frac{1}{2}) = \frac{(2m-1)^3}{3}$. The first moment multiplied by $\sqrt{2}$ of the rectangle $\left\{u,v \mid 0 \leqslant d,\ 0 \leqslant v \leqslant e\right\}$ about the axis $u + v = 0$ is similarly $\frac{de(d+e)}{2} = h(d,e)$ (say). Hence, if the line $x_1 + x_2 = 1$ passes through the center of $N_1 \times C_2$, the first absolute moment multiplied by $\sqrt{2}$ of this rectangle about the axis $x_1 + x_2 = 1$ is (recalling that $b \leqslant m$)

$$J(C_2,N_1) = 2g(b) + 2h(b, \tfrac{m-b}{2}) = \frac{b^3}{3} + \frac{b(m^2-b^2)}{4}.$$ Finally, if

$m \geqslant \frac{1}{2}$ and $N_1 = \left\{x_1 \mid x_1 \leqslant m\right\}$, $C_2 = \left\{x_2 \mid m \leqslant x_2 \leqslant m + b\right\}$, then

$$J(C_2,N_1) = 2g(b) + h(b,1 - b - m) + h(b,2m - 1) = \frac{b^3}{3} + \frac{b}{2}(5m^2 + 3mb - 6m - 2b + 2).$$

We now suppose $\sqrt{A} = k = m + b$ to be fixed. We shall minimize I first under the restriction $m \leq \frac{1}{2}$ (Case I) and then under the restriction $m \geq \frac{1}{2}$ (Case II), and will then compare the two results. We recall $0 \leq b \leq m$ and $\frac{1}{2} \leq k \leq 1$.

<u>Case</u> I. $m \leq \frac{1}{2}$. For any fixed values of m and b, it is evident that one can always move the $N_i$ and $C_i$ around so that $(C_1 \times C_2) \cap Q$ and $(N_1 \times N_2) \cap P$ have zero area and so that the line $x_1 + x_2 = 1$ passes through the center of $N_1 \times C_2$ and $N_2 \times C_1$. This minimizes $J(C_1, N_2)$, $J(C_2, N_1)$, and thus I, since the first absolute moment of any plane figure about an axis which may be moved keeping its slope constant is minimized when the axis divides the figure into two parts of equal area. Thus, for fixed k and b with $k - \frac{1}{2} \leq b \leq \frac{k}{2}$ (i.e., $b \leq m \leq \frac{1}{2}$), the minimum value of $6\,I = 12\,J(C_2, N_1)$ is (see above) $4b^3 - 6b^2k + 3bk^2$. The derivative of the latter with respect to b is $3(k - 2b)^2$, so that in the allowable range of b, I attains its minimum value at $b = k - \frac{1}{2}$, <u>only</u> (i.e., only at $m = \frac{1}{2}$). This value is $I_1(k) = \dfrac{(k-\frac{1}{2})(k^2-k+1)}{6}$ .

<u>Case</u> II: $m \geq \frac{1}{2}$. For fixed m and b, if $b \leq 2 - 3m$ the procedure with $N_i = \left\{x_i \mid 0 \leq x_i \leq m\right\}$ and $C_i = \left\{x_i \mid 1 - \frac{m}{2} - \frac{b}{2} < x_i \leq 1 - \frac{m}{2} + \frac{b}{2}\right\}$ (i.e., such that the line $x_1 + x_2 = 1$ passes through the center of $N_1 \times C_2$ and $N_2 \times C_1$) obviously minimizes all four J's and thus I. The corresponding value of 6 I is $q_k(b) = 6\left\{J(N_1, N_2) + 2J(C_2, N_1)\right\} = 4b^3 + 3bk(k - 2b) + 16(k - b - \frac{1}{2})^3$. This is meaningful only if $k \geq \frac{1}{2}$ and $\max(0, \frac{3}{2}k - 1) \leq b \leq k - \frac{1}{2}$ (i.e., $\frac{1}{2} \leq m \leq \frac{2}{3}$, $b \leq 2 - 3m$, $0 \leq b \leq m$). Its derivative (with respect to b) multiplied by $\frac{1}{3}$ is $(k - 2b)^2 - 4(2k - 2b - 1)^2 = r_k(b)$ (say). This is a concave (downward) parabola

with zeros at $b = \frac{3}{2}k - 1$ and $b = \frac{5}{6}k - \frac{1}{3}$. Since the latter is $\geqslant k - \frac{1}{2}$, $r_k(b)$ is for fixed k positive almost everywhere if $\max(0, \frac{3}{2}k - 1) \leqslant b \leqslant k - \frac{1}{2}$. Thus, in Case II for fixed k and subject to the restriction $b \leqslant 2 - 3m$, $q_k(b)$ (and thus I) is minimized only at $b = \max(0, \frac{3}{2}k - 1)$, the corresponding value of I being

$$I_{21}(k) = \begin{cases} \frac{8}{3}(k - \frac{1}{2})^3 & \text{if} \quad k \leqslant \frac{2}{3}, \\ \frac{k^3}{12} - \frac{(1-k)^3}{3} & \text{if} \quad k \geqslant \frac{2}{3}. \end{cases}$$

It remains under Case II to consider the case where $k > \frac{2}{3}$ (the case $k \leqslant \frac{2}{3}$ being included above) and where b is restricted by $b \geqslant 2 - 3m$ (if $m \geqslant \frac{2}{3}$, this is the only case in Case II). It is now impossible, for fixed b and m (except when $C_i$ is empty), to move the $N_i$ and $C_i$ around so that the line $x_1 + x_2 = 1$ passes through the centers of $N_1 \times C_2$ and $N_2 \times C_1$. However, it is easy to see that, for any fixed m and $b \geqslant 2 - 3m$, the integral $J(N_1, N_2)$ and the integrals $J(C_2, N_1)$ and $J(C_1, N_2)$ are all minimized (and $J(C_1, C_2) = 0$) only if $N_i = \{x_i | x_i \leqslant m\}$ and $C_i = \{x_i | m < x_i \leqslant m + b\}$: the minimization of $J(N_1, N_2)$ is clear, and any other location of the $N_i$ and $C_i$ will move the line $x_1 + x_2 = 1$ even further from the centers of $N_1 \times C_2$ and $N_2 \times C_1$. Thus, for fixed m and $b \geqslant 2 - 3m$, the minimum value of $3I = 3[J(N_1, N_2) + 2J(C_2, N_1)]$ is $8b^3 + 3(4 - 7k)b^2 + 3[(1-k)^2 + (1-2k)^2]b + 8(k - b - \frac{1}{2})^3$. Its derivative (with respect to b) multiplied by $\frac{1}{3}$ is $s_k(b) = 8b^2 + 2(4 - 7k)b + [(1-k)^2 + (1-2k)^2] - 8(k - b - \frac{1}{2})^2$. (The above expressions are meaningful if

$0 \leq b \leq \frac{3}{2}k - 1$, $k > \frac{2}{3}$.) Thus, $s_k(\frac{3}{2}k - 1) = 0$. Moreover, the derivative of $s_k(b)$ with respect to b is $2k > 0$. Hence, $s_k(b)$ is negative for $0 \leq b < \frac{3}{2}k - 1$, so that the minimum value of I subject to $0 \leq b \leq \frac{3}{2}k - 1$ (where $k > \frac{2}{3}$) is $I_{22}(k) = \frac{k^3}{12} - \frac{(1-k)^3}{3}$, which is attained only for $b = \frac{3}{2}k - 1$. Noting that the latter was also the best result under the restriction $b \leq 2 - 3m$ when $k \geq \frac{2}{3}$, we may summarise Case II as follows: the minimum value of I in Case II, attained only when $b = \max(0, \frac{3}{2}k - 1)$, is

$$I_2(k) = \begin{cases} \dfrac{8(k-\frac{1}{2})^3}{3} & \text{if} \quad k \leq \frac{2}{3} \\[2ex] \dfrac{k^3}{12} - \dfrac{(1-k)^3}{3} & \text{if} \quad k \geq \frac{2}{3} \end{cases}.$$

<u>Comparison of $I_1(k)$ and $I_2(k)$</u>: From our results above, we have

$$6\,[I_1(k) - I_2(k)] = \begin{cases} (k - \frac{1}{2})\,(-15k^2 + 15k - 3) & \text{if } \frac{1}{2} \leq k \leq \frac{2}{3} \\[2ex] \frac{3}{2}(1 - k)^3 & \text{if } \frac{2}{3} \leq k \leq 1 \end{cases}$$

Since $15k^2 - 15k + 3 < 0$ for $\frac{1}{2} - \frac{1}{\sqrt{20}} < k < \frac{1}{2} + \frac{1}{\sqrt{20}}$, which includes the range $\frac{1}{2} \leq k \leq \frac{2}{3}$, we see that the setup of Case II is always better than that of Case I (except at $k = \frac{1}{2}$, where they are the same). Thus, for fixed $k = \sqrt{\lambda} \geq \frac{1}{2}$, the minimum of I is given by $I_2(k)$ and is attained <u>only</u> (except for sets of measure zero) by the procedure

$$N_i = \left\{x_i \mid 0 \leq x_i \leq \min(k, 1 - \tfrac{k}{2})\right\}, \quad C_i = \left\{x_i \mid 1 - \tfrac{k}{2} < x_i \leq k\right\}$$

(the latter is empty for $k \leq \frac{2}{3}$).

3. <u>Maximum of</u> cA - I. It remains to find, for any $c > 0$ (the case $c = 0$ is trivial: always phone), the maximum with respect to k (over $\frac{1}{2} \leq k \leq 1$) of $k^2c - I_2(k)$; thus, putting $c = \frac{s}{6}$, we must maximize for each $s > 0$ the function

$$f_s(k) = \begin{cases} k^2 s - 16(k - \frac{1}{2})^3 & \text{if } k \leq \frac{2}{3} , \\ k^2 s - \frac{1}{2} k^3 + 2(1 - k)^3 & \text{if } k \geq \frac{2}{3} . \end{cases}$$

The derivative of $f_s(k)$ with respect to k is

$$f_s'(k) = \begin{cases} 2ks - 48(k - \frac{1}{2})^2 & \text{if } k \leq \frac{2}{3} , \\ 2ks - \frac{3k^2}{2} - 6(1 - k)^2 & \text{if } k \geq \frac{2}{3} . \end{cases}$$

We note that $f_s'$ (but not $f_s''$) is continuous at $k = \frac{2}{3}$ and that $f_s'(\frac{2}{3}) = \frac{4}{3}(s - 1)$. The function $2ks - 48(k - \frac{1}{2})^2$ is a concave (downward) parabola symmetric about $k = \frac{1}{2} + \frac{s}{48}$ and with zeros at $\frac{1}{2} + \frac{s}{48} \pm \frac{1}{48}\sqrt{s^2 + 48s}$. The larger zero, hereafter referred to as $z(s)$, is $\leq \frac{2}{3}$ if $s \leq 1$; the smaller is always $< \frac{1}{2}$. The function $2ks - \frac{3k^2}{2} - 6(1 - k)^2$ is also a concave (downward) parabola, symmetric about $k = \frac{12+2s}{15}$. This function is always negative (there are no zeros) unless $s \geq 3(\sqrt{5} - 2)$, in which case the zeros are at $\frac{12+2s}{15} \pm \frac{1}{15}\sqrt{(12 + 2s)^2 - 180}$. If $s \geq 3(\sqrt{5} - 2)$, the larger zero (say $y(s)$) is always $\geq \frac{2}{3}$ and is $\leq 1$ if $s \leq \frac{3}{4}$; the smaller one (say $y^*(s)$) is always $\leq 1$ and is $\geq \frac{2}{3}$ if $s \leq 1$. At $s = 3(\sqrt{5} - 2)$, the zero is ~~at~~ $\frac{2}{\sqrt{5}}$ .

We summarize the behavior of $f_s'(k)$:

(1) If $s \geq 1$, $f_s'(k)$ is positive (0 at $\frac{2}{3}$ if $s = 1$), so that $f_s(k)$ is maximized at $k = 1$ in this case.

(2) If $\frac{3}{4} \leq s \leq 1$, $f_s'(k)$ is negative for $z(s) < k < y^*(s)$ and non-negative (and positive except for at most three points elsewhere). Hence, the maximum of $f_s(k)$ is either at $z(s)$ or 1.

(3) If $3(\sqrt{5} - 2) < s < \frac{3}{4}$, $f_s'(k)$ is negative for $z(s) < k < y^*(s)$ and $y(s) < k \leq 1$, and is positive elsewhere (except for at most three points). Hence, the maximum of $f_s(k)$ is either at $z(s)$ or $y(s)$.

(4) If $s \leq 3(\sqrt{5} - 2)$, $f_s'(k)$ is positive only for $\frac{1}{2} \leq k < z(s)$; hence, the maximum of $f_s(k)$ is at $k = z(s)$.

It is easy to see that if $k = z(s)$ is the maximizing value of $f_s(k)$ for some s, it is also for all smaller s: as s (and thus, the cost of phoning) decreases, the amount of phoning in the optimum or optimum symmetric policy cannot decrease, as is evident from an examination of the profit of the firm. One may verify by direct computation that $f_s(z(s)) > f_s(1)$ for $s = \frac{3}{4}$, so that throughout (3) the maximum is indeed at $z(s)$. Thus, we need only find the critical value of $s$ ($\frac{3}{4} < s < 1$) for which $f_s(z(s)) = f_s(1)$: i.e., for which

$$2s^4 + 139s^3 - 2016s^2 + 2592s - 864 = 0 \quad .$$

The last equation is satisfied by $s = .773$.

<u>Thus, our final answer for the unique (except for sets of measure zero) optimum symmetric strategy is:</u>

*If* C $\geqslant$ .129:

$$N_i = \left\{ x_i \mid x_i \leqslant \frac{1}{2} \right\} ,$$

$$C_i = \left\{ x_i \mid x_i > \frac{1}{2} \right\} .$$

*If* C $\leqslant$ .129:

$$N_i = \left\{ x_i \mid x_i < \frac{1}{2} + \frac{c}{8} + \frac{1}{8}\sqrt{c^2 + 8c} \right\} \quad \underline{\text{and}}\ C_i\ \underline{\text{empty}}$$

*or*

$$N_i\ \underline{\text{empty}}\ \underline{\text{and}}\ C_i = \left\{ x_i \mid x_i \geqslant \frac{1}{2} - \frac{c}{8} - \frac{1}{8}\sqrt{c^2 + 8c} \right\} .$$

*Either strategy may be used a.* c = .129.