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A Note On The Theory of Forward Markets.

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<u>A Note On The Theory of Forward Markets.</u>[1]

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1. The purpose of this note is to sketch an equilibrium model of futures markets in terms of excess demand functions along lines that have been developed in the theory of equilibrium among spatially connected markets. We shall first discuss the case of perfect certainty about the whole course of the future ("from here to eternity"). This will put the equilibrium problem in its sharpest forms. Thereafter we shall introduce anticipations, which may deviate from the values that will materialize. This will entail considerable modifications of the problems and thus serve to bring out perhaps some of the essentials of "uncertainty". For simplicity we have ignored differences in grades of product in this paper.

The operations of the forward market are reduced here to the following schema: the market convenes at equal time intervals, all transactions taking place there instantaneously. The market price remains unchanged between market times. Clearly, the prices at various times are influenced by and reflect decisions to hold speculative stocks. By these we mean all stocks not tied up in production. (The analogy with Keynes' distinction between the "transaction," "precautionary" and "speculative motive" for holding cash comes to mind.). By perfect certainty or foresight of the market participants we mean not only the absence of precautionary stocks, but also knowledge of the current and all future excess supply functions. At any given time the market does not just fix the present (spot) price and present stocks, but the entire series of future prices and stocks; for without reference to these, the current equilibrium

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[1] I am indebted to H. S. Houthakker for various discussions of the subject and for helpful criticism of the present manuscript.

price cannot be found. Thus the market only needs to come together at time zero to decide the whole future once and for all. This is the situation implied by perfect certainty. Although it seems absurd, it furnishes a useful background to the theory of uncertainty.

2. Let

$t = 0, 1, \ldots T, \ldots$	be (discrete) time points or the subsequent periods,
$p = p(t)$	the spot price at time t, equal to the future's price at all previous times for delivery at time t, if any,
$x = x(t)$	the stock at end of period t,
$c = c(x, t)$	the marginal cost of holding stocks x from time t to t + 1 including interest and depreciation,
$q = q(p, t)$	the excess demand (over current supply) for consumption at time t as a function of the spot price,
$x(0) = a$	the initial stock, and
T	the horizon of the market.

<u>Problem:</u> For given data q, c, a, T to determine the sequence of equilibrium prices and stocks.[2]

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[2] The spatial interpretation is as follows: $c(x, t)$ is the marginal cost of transportation from location t to location t + 1; the costs of transportation in the other direction being prohibitive. q remains the excess-demand function, and the initial stock corresponds to imports at location 0, the final stocks to exports from location T. The equilibrium of market systems of this kind is treated e.g. in Samuelson [1].

Note: It is not necessarily assumed that q shows seasonality or is such as to render stock holdings unfeasible over longer than a certain period. This makes the model applicable to all kinds of futures' markets and also to such speculative markets as of securities.

3. The following necessary relations between stocks and price differences (spreads) are intuitive.

The "equation of continuity":

(1) $x(t) - x(t-1) + q(p(t), t) = 0;$

and the "price gradient equation":

(2) $p(t+1) - p(t) \left\{ \begin{matrix} = \\ \leq \end{matrix} \right\} \quad c(x(t), t) \quad \text{if } x(t) \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0.$

Our assertion is, that together with the initial condition

(3) $x(0) = a$

and the terminal condition (which, though intuitive, can be derived explicitely, as seen below)

(4) $x(T) = 0$

the necessary relations (1) ... (4) are also sufficient to determine all prices and stocks for $t = 0, 1, \ldots T$. To demonstrate this we first show that (1) and (2) may be compressed into a single equation which is the solution equation of the problem of maximizing "social payoff" (Samuelson) or net consumer's surplus with respect to the stocks held.

Let $p = \pi(q, t)$ denote the inverse function of $q = q(t, p)$.

Then (1) becomes

$p(t) = \pi(q(t), t) = \pi(x(t) - x(t-1), t)$

Inserting this into (2) yields

$$(5)\ \pi(x(t+1) - x(t),\ t+1) - \pi(x(t) - x(t-1),\ t) \left\{ \begin{matrix} = \\ \leq \end{matrix} \right\} c(x(t),t) \text{ if } x(t) \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0,$$

a second-order difference equation (inequality).

4. Consider now

$$S_T = \underset{x(t) \geq 0}{\text{Max}} \sum_{t=1}^{T} \left[\int_0^{x(t-1) - x(t)} (q,\ t)\ dq - \int_0^{x(t)} c(u,\ t)\ du \right]$$

which is the net of consumers' surpluses, allowance being made for storage costs. A necessary condition for it to be maximized is that all derivatives with respect to the $x(t)$ $t = 0, \ldots T$ be non-positive, and be zero if the variable $x(t)$ itself is positive. For $t = 0, \ldots T - 1$ this yields precisely the set of equations and inequalities (5) and for $t = T$ the terminal condition (4)

From the equation system (1) ... (4) itself it may be shown that its solution (if it exists) is unique. (This is done in the usual way by considering

an expression $$0 = \sum_{t=1}^{T} (p_1(t) - p_2(t)) [x_1(t) - x_1(t-1) - x_2(t) + x_2(t-1) + q(p_1,\ t) - q(p_2,\ t)]$$ transforming and observing that the $c(x,\ t)$ are non-decreasing and the $q(p,\ t)$ are non-increasing functions of x, p respectively. A proof via joint concavity of S_T in the $x(t)$ does not seem possible).

It follows now that (4) (5) and the initial condition (3) are necessary and sufficient for the solution of the maximum problem, hence that a unique solution of (1), (2), (3), (4) exists which is the equilibrium set of stocks and prices.

5. How does one find the solution to this system?

With initial stock given, assume a tentative initial price (for t=1). Then (1) determines the stock at the end of this period which is equal to that at the beginning of the next one. So long as stocks are positive, the price for the next period is determined by (2). Continuing in this way until stocks are zero and assuming a new tentative price one proceed to T. By adjusting the assumed opening prices for each series so as to satisfy the terminal condition and avoid contradiction of the inequality in (2) at the beginning of series, we are sure to obtain a solution, which we know also to be the only solution.

6. Suppose now that T is not the horizon, but just one specific time point, the market horizon being infinite. Then stocks at time T are not necessarily zero any more but, say, equal to b. It is clear that (1), (2) remain necessary conditions with $S(0) = a$, $S(T) = b$ as boundary conditions. How are we to interpret the former condition (4)?

It might seem, that the system is now indeterminate. For, as one easily verifies, the solution $x_T(t)$, $p_T(t)$ as a function of the horizon T, does not in general converge as T goes to infinity. This however does not settle the existence problem for solutions to (1), (2), (3), (4), rather it makes uniqueness doubtful. [The usual uniqueness proof indeed does not work any more].

A solution may indeed be constructed in the former way. Only this will require infinitely many steps which cannot be reduced to a finite procedure if the $q(p, t)$ are really independent, i.e. subject to no law of regularity. The requirement, that terminal stocks be zero takes here the form that it be possible for terminal stocks to be zero, namely, that no accumulation of stocks take place beyond all limits. Thus (4) reduces to a condition, that the limit of some average of $x(t)$ remain finite. The vagueness of this condition is to some extent compensated for by the sensitivity of stocks to the price chosen initially, which increases with proceeding time. Thus the finiteness condition (4) may indeed limit the set of initial prices to a small range.

7. Analytically one may ask the following questions: Given a known time series of $q(p, t)$, e.g. a periodical one, what is the series of $x(t)$, $p(t)$? In the periodical case one expects of course the prices and stocks to be periodical, too. Here only a finite calculation is necessary.

Another question is, given the probability distribution of the time series of q, what is the probability distribution of x, p at a fixed time or at any time? Here it would be necessary of course to make some drastic simplifications, e.g. take q, c to be linear functions of p and x, respectively and restrict the time series of the parameters of these functions which contain t, to classes that are well known (stationary, Markoffian processes, etc.). Even in the simplest cases, the presence of the inequality sign in (2) is apt to cause difficulties for the analytic treatment.[3]

8. The determination of present prices and stocks from the model may be considered a "finalistic" one: the chain of reasoning runs toward and back from the infinite future. This is, in a way, opposed to the

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[3] To circumvent this one may think of introducing a negative cost of storage for sufficiently small stocks that goes to infinity as x reaches zero. But except for rather arbitrary cases of assumptions this only shifts the trouble to that of handling a piecewise defined storage cost function $c(x)$.

causality principle, in which the past is the only determinant. Mathematically this difference is reflected in the fact, that while causation makes the solution depend only on initial condition, the present, finalistic approach uses boundary condition, one of which is located in the infinite future. It would be futile of course to explain the sequence of prices and stocks in a future's market from considerations whose realization would itself require an infinite time (for an infinite number of steps). Even if the future were perfectly knowable, the market would have to have recourse to an approximation scheme, based ultimately on the experiences of the past. This lends theoretical justification to replacing the system (1), (2) by a causalistic one. By disregarding part of the information that would otherwise be obtained from the infinite future, the problem necessarily involves some uncertainty now.

For the prices, which have been projected from the past may at any time prove too high, in which case stocks tend to accumulate, or too low, in which case an effective current scarcity arises. Thus prices are subject to revision at any time, and this means uncertainty.

9. In terms of our model, the price gradient equation (2) must now be replaced by a "price formation equation" which relates the current price to the past history. Since (1) in effect reduces stocks to a function of prices, we may eliminate stocks from the history and insert instead the parametersof the past and current excess demand functions. Let

$q^o(t)$ be the vector of parameters which characterises the excess demand function at time t. Furthermore, let $e(t)$ be a vector of currently available observations pertinent to the forecast of the future excess demand function(s). Thus

(6) $$f[e(t), q^o(t), q^o(t-1), \ldots\ q^o(t- \ )\ \ldots ;\ p(t-1), p(t-2), \ldots] = p(t)$$

is the form in which we assume the price-formation equation to be given.

In fact, (1) has now become redundant, since it is implicit in (6). The movement of prices appears now as determined only by the consecutive observations of actualized excess demand functions. To complete the picture of a future market we have only to assume this price formation equation to be given with reference to a series of future dates.

Let $p_\tau(t)$ be the price at time t of future delivery at time $t + \tau$. (p_o is then the spot price). Then we have a system

$$(7) \quad f_\tau[e(t)\, q^o(t),\ q^o(t-1),\ \ldots\ ;\ p_o(t-1),\ p_o(t-2),\ \ldots\ ] = p_\tau(t)$$

$$\tau = 1,\ \ldots.\ T$$

determining all prices, where T now denotes the "longest future". The mathematical form of the function f_τ is taken to be known. We may assume that the market has been running long enough for the price expectation functions to have become stabilized. That is to say, the best method of learning from the past have been adopted and technological progress in that direction is excluded.

Note that (7) is not a system of "simultaneous" but of independent equations.[4/] Note also that (1) may be used to infer the "implied" expected stocks. For the futures' prices must consistently satisfy relation (2).

10. If there are several, spatially connected, forward markets, we may easily generalize the relations (1), (2) of perfect certainty. Let $\Delta_{ij}t$ denote the time required for transportation of stocks from the

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4/ But it becomes a system of simultaneous equations if the spread between spot and futures' prices gives rise to substitution between current and future consumption for then the equations (7) must be changed so as to contain all of the prices $p_\tau(t)$.

market at location i to that at location j, and let k_{ij} be transportation cost, $x_i(t)$ the stocks at i and $x_{ij}(t)$ the flows from i to j.

$$\text{(1a)} \quad x_i(t) - x_i(t-1) + \sum_j [x_{ij}(t) - x_{ji}(t)] + q_i\,(p_i(t),\, t) = 0$$

which says that excess supply (the negative of excess demand) equals net accumulation of stocks plus net exports. Similarly we have the price gradient conditions

$$\text{(2a)} \quad p_j(t + \Delta_{ij}t) - p_i(t) \left\{\begin{matrix} = \\ \leq \end{matrix}\right\} k_{ij} \qquad \text{if } x_{ij}(t) \left\{\begin{matrix} > \\ = \end{matrix}\right\} 0$$

$$\text{(2b)} \quad p_i(t+1) - p_i(t) \left\{\begin{matrix} = \\ \leq \end{matrix}\right\} c_i(x_i(t),\, t) \quad \text{if } x_i(t) \left\{\begin{matrix} > \\ = \end{matrix}\right\} 0$$

Here it has been assumed that the forward market actually distinguishes between delivery at various locations. If this is not so, then the futures prices should of course be equal among the various market locations. But this would be inconsistent with the assumption of our analysis, that the present future's prices equal the future spot prices.
Again we have boundary conditions

$$\left.\begin{matrix} \text{(3a)} \quad x_i(0) = a_i \\ \text{(3b)} \quad x_i(T) = 0 \end{matrix}\right. \qquad \text{for all } i.$$

It is not difficult to construct the "social surplus" function and show uniqueness of prices and stocks, and of flows up to "neutral circuits" [2]. Because of the spatial interdependence of markets a trial and error constructions of the solution is much more difficult now. But the principles of determination remain unchanged. If the excess demand functions are periodic, one may conclude that the prices, stocks and flows too show a periodic pattern. The possibility of exchange among spatially connected markets has of course the tendency of reducing the fluctuations of price, and hence stocks, in each market.

11. If we now assume again imperfect knowledge of the future excess demand functions, we must introduce price formation equations.

(7a) $f_{i,\tau}(e(t), q^o(t), q^o(t-1), \dots ; p_{i,o}(t-1), p_{i,o}(t-2) \dots ,$
$p_{j,o}(t-1) \dots) = p_{i,\tau}(t).$

This assumes again that the market distinguishes between future delivery at the various locations. Otherwise the futures' prices are all equal, and as the date of delivery draws near must approximate the spot price at that location where delivery is most economical for the party that exercises the option. Spot prices of course do not tend to be equal.

However, since the unit costs of transportation are here assumed independent of the flows, the future's prices must satisfy certain compatibility conditions, for otherwise it would pay to make shipments in specie.

(2c) $$p_{j,\tau}(t) - p_{i,\tau}(t) \left\{ \begin{matrix} = \\ \\ \leq \end{matrix} \right\} k_{ij}$$

provided of course τ exceeds the time required for shipment.

The latter restriction exempts the spot prices from following the same pattern. - If the marginal costs of holding stocks are independent of the quantity of stocks (provided they are positive), a similar precise restriction would have to hold for the spreads between the various future's prices, including the spot price.

12. To summarize, the theory of equilibrium in forward markets appears to require considerations of a nature essentially different from those that afford insight into the interdependence of spatially connected markets, even if the future were perfectly knowable, the essential difficulty being the infinity of time. While the "equation of continuity" thus lacks operational significance for future stocks, the "price-gradient equation" remains partly effective by placing limits on the spreads among the (present) futures' prices. If the market distinguishes between locations of delivery, the locational spread is similarly limited by (anticipated) transportation costs.

It seems then, that any attempt at explaining the series of prices and stocks in forward markets be better adressed to the formation of anticipations in the market from past history and current observations, rather than to the structure of an infinite series of excess demand functions.

REFERENCES

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[2] KOOPMANS, T. C. and REITER, S. A Model of Transportation, Cowles Commission Monograph 13, p. 248.